June 22, 2000

Quantum BRST properties of reparametrization invariant theories.

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Abstract

Any regular quantum mechanical system may be cast into an abelian gauge theory by simply reformulating it as a reparametrization invariant theory. We present a detailed study of the BRST quantization of such reparametrization invariant theories within a precise operator version of BRST. The treatment elucidates several intricate aspects of the BRST quantization of reparametrization invariant theories like the appearance of physical time. We propose general rules for how physical wave functions and physical propagators are to be projected from the BRST singlets and propagators in the ghost extended BRST theory. These projections are performed by boundary conditions which are precisely specified by the operator BRST. We demonstrate explicitly the validity of these rules for the considered class of models. The corresponding path integrals are worked out explicitly and compared with the conventional BFV path integral formulation.

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1 Introduction.

The main purpose of the present paper is to obtain a precise understanding of the quantum properties of reparametrization invariant theories as they appear in a BRST quantization. We want to know precisely how physical objects may be extracted and in particular how physical time appears from the formalism. We restrict ourselves to the framework of ordinary quantum mechanics with finite number of degrees of freedom since we are interested in the generic case without topological obstructions.

That any regular classical Lagrangian mechanical system may be cast into an equivalent reparametrization invariant theory has been known for long [1, 2]. The procedure is as follows: Consider a regular Lagrangian theory with the Lagrangian $L(t) \equiv L(\dot{q}(t), \dot{q}(t); t)$. ($L(t)$ is regular if $\det \frac{\partial^2 L(t)}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$.) Consider then the action and replace time t by an arbitrary parameter τ:

$$\int dt L(t) = \int d\tau\, \dot{t} L(t(\tau)), \quad \dot{t} \equiv \frac{dt}{d\tau} > 0. \tag{1.1}$$

The new Lagrangian $L'(\tau) \equiv \dot{t}L(t(\tau))$ does then obviously describe the same theory. However, $L'(\tau)$ contains time $t(\tau)$ as a new degree of freedom and is singular. Due to the invariance under reparametrizations $\tau \rightarrow \tau'(\tau)$ we have the constraint

$$\pi + H(t) \equiv \frac{\partial L'}{\partial \dot{t}} + H(t) = 0, \tag{1.2}$$

where $H(t)$ is the Hamiltonian of the original theory described by $L(t)$. $H(t)$ depends explicitly on time when $L(t)$ does. In this way we have turned the original regular theory into an abelian gauge theory with the constraint variable $\pi + H(t)$ as gauge generator. In appendix A we derive all relevant observables, *i.e.* all relevant gauge invariant objects of this classical gauge theory. The physical time is shown to be just the gauge fixed time.

In a Dirac quantization of the above reparametrization invariant theory the Schrödinger equation appears as a pure constraint equation. The equivalence with the original theory is then obvious. (For a recent application see [3].) The gauge invariant variables in appendix A may be turned into operators by means of which formal solutions and propagators of the Schrödinger equation may be constructed. (A corresponding detailed treatment of the relativistic particle with and without spins were given in [4].)

In the present paper we perform a BRST quantization of the above reparametrization invariant theory. We call it cohomological quantum mechanics to distinguish it from the quantization of the original Lagrangian. It has been partly treated in *e.g.* [2, 5, 6]. However, here we perform a very detailed analysis of this BRST quantization based on a precise operator formulation on inner product spaces, a formulation which also was used in [6]. We shall explicitly demonstrate the relation to ordinary quantum mechanics which is much more involved than in the Dirac quantization above. In the process we also find the necessary ingredients of the BRST theory for this correspondence to exist. We consider general gauge fixings and general state representations. In particular, we discuss in detail how time appears from the reparametrization invariant version. (The problems with physical time in reparametrization invariant theories, particular gravity, is discussed in [7].) In addition we propose general projection formulas for physical wave functions and propagators, which for cohomological quantum mechanics may be directly compared with the corresponding objects in the original quantum theory.

The paper is organized as follows: In section 2 we present the precise operator version of the BRST quantization which we then use for cohomological quantum mechanics in section 3. In section 4 we consider possible wave function representations of a set of BRST singlets. We propose general formulas for physical wave functions and propagators which are shown to yield the expected results for this set. In section 5 we consider another set and discuss the differences. In section 6 we give and calculate path integral expressions for propagators, compare them with the conventional BFV formulation and show how they may be projected. Finally we conclude the paper in section 7. In appendix A we give gauge invariant extensions for the classical version of the reparametrization invariant theory. In appendix B we give some properties of our ghost states and in appendix C we apply our projection formulas to the relativistic particle.

2 Our BRST approach.

There are several roads to a satisfactory quantization of general gauge theories (see [2] for a review). The general approach is called BRST quantization which is cohomological in nature. There are two versions. One is Hamiltonian and is called BFV-BRST quantization [8], and the other is Lagrangian and is called field-antifield BV-quantization [9]. Both were originally designed for path integral quantizations. In the BFV-BRST approach one identifies the classical constraints and introduces generalized Faddeev-Popov ghosts to construct a nilpotent odd BRST charge in a Poisson sense. BFV has given a general prescription for this charge. They have also a general prescription for the BRST invariant Hamiltonian which has to be inserted into a phase space version of the path integral. The BV-approach on the other hand has a completely different form but should yield an equivalent result at the end. Both these methods yield an effective theory. However, to handle these effective theories one has often to do some modifications due to the necessity to work on inner product spaces. In quantum field theory one typically has to consider a Euclidean version of the effective theory.

In the present paper we shall make use of a precise operator formulation of BRST quantization on inner product spaces based on the BFV scheme. It allows for a detailed treatment of the quantum theory and provides for algorithms for the physical states. Any operator BRST quantization requires [10]

$$Q|ph\rangle = 0, \quad Q^2 = 0, \tag{2.1}$$

where $|ph\rangle$ is any physical state, and where Q is the odd, hermitian BRST charge operator which lives in a ghost extended framework. In this framework the operators and states may be decomposed into eigenoperators and eigenstates to the ghost number operator N. In particular we have $[N, Q] = Q$, which means that Q has ghost number one. To construct a nilpotent Q one may use the general BFV-prescription for the pseudoclassical Q and then quantize. For finite degrees of freedom one always arrives in this way at a nilpotent operator Q by means of an expansion in $\hbar$ [11]. However, for infinite degrees of freedom such a solution may not exist in which case we have an anomalous gauge theory. Infinite degrees of freedom also requires a regularization in the calculations. These difficulties will not appear here since we only consider finite degrees of freedom in the following.

The condition that the original state space is an inner product space leads to a more precise algorithm than (2.1) and provides for a direct connection to the BFV path integral

formulation. A first algorithm was given in [12, 13, 14] where it was proposed that the BRST charge operator, Q, must be possible to decompose as follows (we use graded commutators)

$$Q = \delta + \delta^\dagger, \quad \delta^2 = 0, \quad [\delta, \delta^\dagger] \equiv \delta\delta^\dagger + \delta^\dagger\delta = 0, \tag{2.2}$$

in which case the inner product solutions of (2.1) are the solutions of

$$\delta|ph\rangle = \delta^\dagger|ph\rangle = 0. \tag{2.3}$$

(The decomposition (2.2) requires dynamical Lagrange multipliers and antighosts in general.) The decomposition (2.2) and $|ph\rangle$ are expected to be unique up to BRST invariant unitary transformations in topological trivial theories. The solutions of the conditions (2.3) obtained so far turns out to have the general form [14, 15, 16, 17, 18]

$$|ph\rangle = e^{[Q,\psi]}|\phi\rangle, \tag{2.4}$$

where ψ is a hermitian, odd gauge fixing operator with ghost number minus one, and where $|\phi\rangle$ is a BRST invariant state determined by a set of simple hermitian operators. (The inner product of (2.4) is manifestly real.)

In a second approach to BRST quantization on inner product spaces the BRST invariant inner product states are from the very beginning assumed to be of the form (2.4). The $|\phi\rangle$-states are then required to satisfy the following conditions [16, 18]

$$B_i|\phi\rangle = 0, \quad B_i = a[Q, C_i] \tag{2.5}$$

where a is a c-number, and where B_i and C_i are hermitian operators which are in involutions. ($C_i|\phi\rangle = 0$ are then allowed gauge fixing conditions [19, 16].) The index i runs over one-fourth of the number of independent unphysical operators. (Conditions (2.5) imply then $Q|\phi\rangle = 0$.) However, these conditions are not sufficient to make the physical states (2.4) inner product states. (2.5) is similar to a Dirac quantization and it is well-known that an odd B leads to a zero norm state while an even B leads to a state with infinite norm if the spectrum of B is continuous. In order for the zeros and infinities to cancel we must have an equal number of even and odd B_i-operators. This requires the total number of B_i-operators to be even. (This is always the case in the presence of dynamical Lagrange multipliers and antighosts.) However, the inner product of the $|\phi\rangle$-state in (2.4) is still undefined without an appropriate regulator factor $e^{[Q,\psi]}$. The precise condition for the hermitian gauge fixing fermion is [16]

$$[D'_r, (D'_s)^\dagger] \text{ is an invertible matrix operator} \tag{2.6}$$

where

$$D'_r \equiv e^{[Q,\psi]} D_r e^{-[Q,\psi]} \tag{2.7}$$

where in turn $D_r \equiv \{B_i, C_i\}$ are the hermitian BRST doublets in involution. Condition (2.6) implies that D'_r and $(D'_r)^\dagger$ form generalized BRST quartets [10].

The conditions (2.5) and (2.3) are connected if the δ operator in (2.2) is of the form

$$\delta = A\Big(e^{[Q,\psi]} B_a e^{-[Q,\psi]}\Big)^\dagger \Big(e^{[Q,\psi]} B'_a e^{-[Q,\psi]}\Big), \tag{2.8}$$

where B_a and B'_a are the bosonic and fermionic parts of the B_i-operators ($B_i = \{B_a, B'_a\}$). The conditions (2.3) yield then always (2.4) where $|\phi\rangle$ satisfies (2.5) if A is a bosonic factor which commutes with the other factors in δ. Since the expression (2.8) does not satisfy the properties (2.2) for all possible B-operators and all possible gauge fixings fermions ψ, the relation between the two approaches is unclear. However, for the models considered so far there is no restriction on the physics to require (2.8) to satisfy (2.2). One may notice that the choice of B_i-operators in (2.5) is not unique. There are unitary equivalent sets where the unitary operators are BRST invariant. In addition there are different sets which yields the same restricted $|\phi\rangle$ which will be exemplified in the next section.

The inner product of the states (2.4) will in the path integral formulation include the form proposed by BFV, and the conditions on $|\phi\rangle$ appear then as boundary conditions [20] as will also be demonstrated in section 6. However, there are some technical points in this correspondence which have to do with the reality properties of the effective Lagrangian. BRST quantization requires half of the unphysical variables to be quantized with indefinite metric states which means that half of the basic hermitian unphysical operators must have imaginary eigenvalues. This implies that the effective Lagrangian from the operator version is not real in general. However, it leaves a certain freedom in the choice of quantum states which means that it produces several possible choices of effective Lagrangians. There is no hint of these choices within the path integral formulation off hand.

When the gauge theory contains a nontrivial Hamiltonian the above operator quantization on inner product states is not well defined in general. The only well defined general approach seems then to be to reformulate the original theory as a reparametrization invariant one. The reason is that only in a reparametrization invariant theory is the Hamiltonian trivial (zero) and we may rely on the previous formulation. In the resulting framework we may then forget about the Hamiltonian and only worry about one nilpotent BFV-BRST charge operator now containing the Hamiltonian itself [6]. This procedure also applies to ordinary quantum mechanics in which case one finds just cohomological quantum mechanics which will be treated here.

3 Formal BRST solutions of cohomological quantum mechanics.

In the introduction we reviewed how a theory described by a regular Lagrangian $L(t)$ may be replaced by the singular reparametrization invariant Lagrangian $L'(\tau) \equiv L(t(\tau))\dot{t}$. As the original theory given by $L(t)$ may be equivalently described by the first order phase space Lagrangian $L_0(t) = p_i\dot{q}^i - H(t)$, where q^i and p_i are the phase space variables, also the reparametrization invariant theory $L'(\tau)$ may be described by an equivalent first order phase space Lagrangian. It is given by

$$L_1(\tau) \equiv p_i\dot{q}^i + \pi\dot{t} - v(\pi + H(t)), \tag{3.1}$$

where v is a Lagrange multiplier. The original regular theory is then turned into an abelian gauge theory with the two constraints $\pi + H(t)$ and $\pi_v = 0$, where π_v is the conjugate momentum to the Lagrange multiplier v.

We quantize now the reparametrization invariant theory $L'(\tau)$ as described by $L_1(\tau)$. The original variables, $p_i, q^i, \pi, t, \pi_v, v$ are then turned into operators which we denote by

capital letters. The nontrivial part of their commutator algebra is ($\hbar = 1$)

$$[Q^i, P_j] = i\delta^i_j, \quad [T, \Pi] = i, \quad [V, \Pi_V] = i. \tag{3.2}$$

Following the BFV-prescription for the BRST charge we find the odd, nilpotent operator

$$Q = \mathcal{C}(\Pi + H(T)) + \bar{\mathcal{P}}\Pi_V, \tag{3.3}$$

where we have introduced the odd, hermitian ghost operators $\mathcal{C}, \bar{\mathcal{C}}$ and their conjugate hermitian momenta $\mathcal{P}, \bar{\mathcal{P}}$. ($\bar{\mathcal{C}}$ is an antighost.) Their nontrivial commutators are

$$[\mathcal{C}, \mathcal{P}] = 1, \quad [\bar{\mathcal{C}}, \bar{\mathcal{P}}] = 1. \tag{3.4}$$

$H(T)$ in (3.3) is the hermitian Hamiltonian operator corresponding to the original classical Hamiltonian in (1.2) and (3.1). In [18] we made a rather complete investigation of simple abelian models with linear gauge fixings in which case it was explicitly demonstrated that (2.2)-(2.3) lead to solutions of the form (2.4).

We look now directly for solutions of the form (2.4). First we need a BRST invariant state $|\phi\rangle$ determined by linear hermitian constraints of the form $B_i|\phi\rangle = 0$ where $B_i = [Q, C_i]$. In the present case we have eight unphysical operators which means that we have two hermitian B_i-operators, one odd and one even according to the rules given in the previous section. One allowed choice of conditions is

$$\Pi_V|\phi\rangle = \mathcal{C}|\phi\rangle = 0 \quad \Leftrightarrow \quad [Q, \bar{\mathcal{C}}]|\phi\rangle = [Q, T]|\phi\rangle = 0. \tag{3.5}$$

Another is

$$\bar{\mathcal{P}}|\phi\rangle = \Big(\Pi + H(T)\Big)|\phi\rangle = 0 \quad \Leftrightarrow \quad [Q, \mathcal{P}]|\phi\rangle = [Q, V]|\phi\rangle = 0. \tag{3.6}$$

Both these choices imply $Q|\phi\rangle = 0$. Note, however, that

$$\mathcal{C}|\phi\rangle_0 = \bar{\mathcal{P}}|\phi\rangle_0 = 0, \tag{3.7}$$

or

$$\Pi_V|\phi\rangle_0 = \Big(\Pi + H(T)\Big)|\phi\rangle_0 = 0, \tag{3.8}$$

are not allowed although they imply $Q|\phi\rangle = 0$. Eq.(3.7) makes $|\phi\rangle$ have ghost number plus one which implies that $|ph\rangle$ in (2.4) is a zero norm state for whatever choice of ψ. For (3.8) on the other hand there exists no choice of ψ such that the corresponding $|ph\rangle$ is an inner product state with finite norm. Only (3.5) and (3.6) contain one even and one odd B-operator. They also determine the allowed boundary conditions in the path integral formulation as will be seen in section 6. (Boundary conditions corresponding (3.5) have *e.g.* been considered by Henneaux and Teitelboim [2, 21].)

In abelian gauge theories all allowed linear conditions on $|\phi\rangle$ are related by means of extended unitary gauge transformations of the form [18]

$$|\phi\rangle' \equiv U|\phi\rangle, \quad U = \exp\{i[Q, \rho]\}, \tag{3.9}$$

where ρ is an odd, hermitian operator with ghost number minus one. (U has then ghost number zero and does not affect the ghost number properties of the states.) By means of U one may go to a unitary equivalent basis of operators,

$$\tilde{A} = UAU^\dagger, \tag{3.10}$$

where A is any operator. In particular we have $\tilde{Q} = Q$. From (3.5) we find then that the $|\phi\rangle'$-state (3.9) satisfies

$$\tilde{\Pi}_V|\phi\rangle' = \tilde{\mathcal{C}}|\phi\rangle' = 0, \tag{3.11}$$

if *e.g.* $|\phi\rangle$ satisfies (3.5). This represents essentially all allowed choices of $|\phi\rangle$ in (2.4) as well as all allowed boundary conditions in the BFV path integral formalism here. For instance, the special conditions (3.6) are included in (3.11). To see this it is sufficient to consider the following special U (an abelian subgroup of (3.9))

$$\begin{aligned} &U(\theta) = \exp\{i[Q, \rho(\theta)]\} = \exp\Big\{\theta\Big(V(\Pi + H(T)) - T\Pi_V + i\mathcal{P}\bar{\mathcal{P}} - i\bar{\mathcal{C}}\mathcal{C}\Big)\Big\},\\ &\rho(\theta) \equiv \theta(\mathcal{P}V - \bar{\mathcal{C}}T), \end{aligned} \tag{3.12}$$

where θ is a real parameter. The conditions (3.11) become then explicitly ($|\phi\rangle_\theta \equiv U(\theta)|\phi\rangle$)

$$\begin{aligned} &\Big(\Pi_V \cos\theta - (\Pi + H(T))\sin\theta\Big)|\phi\rangle_\theta = 0,\\ &\Big(\mathcal{C}\cos\theta + \bar{\mathcal{P}}\sin\theta\Big)|\phi\rangle_\theta = 0. \end{aligned} \tag{3.13}$$

Thus, for $\theta = 0$ we have (3.5) while for $\theta = \pi/2,\ 3\pi/2$ we have (3.6).

Now the inner product of the $|\phi\rangle$-states (3.5) or (3.6) are not well defined. We need a gauge fixing factor $e^{[Q,\psi]}$ where ψ is an odd, hermitian gauge fixing operator according to the prescription (2.4). For the $|\phi\rangle$-state (3.5) we may choose the physical inner product solution to be

$$|ph\rangle = e^{[Q,\psi]}|\phi\rangle, \quad \psi = \mathcal{P}\Lambda(V), \tag{3.14}$$

which is satisfactory if $\Lambda'(V) > 0$ or $\Lambda'(V) < 0$. The simple argument for this is that $\Lambda(V)$ is a gauge fixing operator to Π_V which annihilates $|\phi\rangle$. The precise argument follows from the basic criterion (2.6) (see below). Similarly we may choose

$$|ph\rangle = e^{[Q,\psi]}|\phi\rangle, \quad \psi = \bar{\mathcal{C}}\chi(T), \tag{3.15}$$

for the $|\phi\rangle$-state (3.6). This is satisfactory if $\dot{\chi}(T) > 0$ or $\dot{\chi}(T) < 0$. There are of course more general forms allowed for ψ. In particular, we may choose forms which are good for whatever choice of conditions on $|\phi\rangle$. A natural choice is then

$$\psi = \mathcal{P}\Lambda(V) + \bar{\mathcal{C}}\chi(T), \tag{3.16}$$

which is in accordance with the BFV prescription for path integrals. In fact, this ψ reduces to the ones in (3.14) and (3.15) at least for linear Λ and χ. Thus, the conditions on $|\phi\rangle$ makes a whole class of gauge fixing fermions equivalent. Other forms of ψ are obtained by extended unitary gauge transformations of the form (3.9) which does not affect the conditions on $|\phi\rangle$ [18].

It is straight-forward to prove that (3.14) is an inner product solution by means of the basic criterion (2.6). The set of hermitian BRST doublets in involutions are here $D_r \equiv \{\Pi_V, \mathcal{C}, \chi(T), \bar{\mathcal{C}}\}$ where $\chi(T)$ and $\bar{\mathcal{C}}$ are hermitian gauge fixing variables. (We may impose the conditions $\chi(T)|\phi\rangle = \bar{\mathcal{C}}|\phi\rangle = 0$ provided $\dot{\chi}(T) \neq 0$ since $[Q, \chi(T)] = \mathcal{C}\dot{\chi}(T)$ must be equivalent to $\mathcal{C}$ on states.) The corresponding transformed nonhermitian doublets defined by (2.7) are then explicitly (assuming $\acute{\chi} = \chi(\acute{T})$)

$$\begin{aligned}
&\acute{\Pi}_V \equiv e^{[Q,\psi]}\Pi_V e^{-[Q,\psi]} = \Pi_V + i(\Pi + H(T))\Lambda'(V) + \bar{\mathcal{P}}\mathcal{P}\Lambda''(V),\\
&\acute{\mathcal{C}} \equiv e^{[Q,\psi]}\mathcal{C}e^{-[Q,\psi]} = \mathcal{C} - i\bar{\mathcal{P}}\Lambda'(V),\\
&\chi(\acute{T}) \equiv \chi(T - i\Lambda(V)),\\
&\acute{\bar{\mathcal{C}}} \equiv \bar{\mathcal{C}} + i\Lambda'(V)\mathcal{P}.
\end{aligned} \tag{3.17}$$

The only nonzero elements of the matrix operator $[\acute{D}_r, (\acute{D}_s)^\dagger]$ are then

$$\begin{aligned}
&[\acute{\mathcal{C}}, (\acute{\Pi}_V)^\dagger] = 2\Lambda''(V)\bar{\mathcal{P}}, \quad ,[\acute{\bar{\mathcal{C}}}, (\acute{\Pi}_V)^\dagger] = -2\Lambda''(V)\mathcal{P},\\
&[\chi(\acute{T}), (\acute{\Pi}_V)^\dagger] = 2\dot{\chi}(\acute{T})\Lambda'(V), \quad [\acute{\mathcal{C}}, (\acute{\bar{\mathcal{C}}})^\dagger] = -i2\Lambda'(V).
\end{aligned} \tag{3.18}$$

Hence, a necessary condition for the matrix operator $[\acute{D}_r, (\acute{D}_s)^\dagger]$ to be nonsingular is that $\Lambda'(V) \neq 0$ as asserted above ($\dot{\chi} \neq 0$). Note that $|ph\rangle$ in (3.14) is a solution to

$$\acute{\Pi}_V|ph\rangle = \acute{\mathcal{C}}|ph\rangle = 0 \tag{3.19}$$

due to (3.5). However, the rules above also allow for the dual conditions

$$(\acute{\Pi}_V)^\dagger|ph'\rangle = (\acute{\mathcal{C}})^\dagger|ph'\rangle = 0, \tag{3.20}$$

with the solution

$$|ph'\rangle = e^{-[Q,\psi]}|\phi\rangle, \quad \psi = \mathcal{P}\Lambda(V). \tag{3.21}$$

This solution is equivalent to (3.14) with $\psi = -\mathcal{P}\Lambda(V)$. The two solutions (3.14) and (3.21) are related by $\Lambda \leftrightarrow -\Lambda$ and correspond to the two possibilities $\Lambda'(V) > 0$ and $\Lambda'(V) < 0$.

The BRST charge (3.3) may be decomposed according to (2.2) with δ given by

$$\delta = \frac{i}{2\Lambda'(V)}(\acute{\Pi}_V)^\dagger\acute{\mathcal{C}}, \tag{3.22}$$

provided $\Lambda'(V)$ is a nonzero constant, *i.e.* provided $\Lambda(V)$ is linear in V. (We do not know how the decomposition (2.2) looks like for $\Lambda''(V) \neq 0$.) Since $\acute{\mathcal{C}}$ and $(\acute{\Pi}_V)^\dagger$ commute when $\Lambda''(V) = 0$ the conditions (2.3) lead to either the solution (3.14) or (3.21). A restriction to a linear $\Lambda(V)$ does not affect the physical content of the theory as we shall see.

All other inner product solutions than those considered above are obtained by means of gauge transformations represented by unitary operators of the form (3.9). These solutions are

$$|ph\rangle' = U|ph\rangle = e^{\pm[Q,\tilde{\psi}]}|\phi\rangle', \quad \tilde{\psi} = U\psi U^\dagger, \tag{3.23}$$

where $|\phi\rangle'$ is given by (3.9). That U generates gauge transformations on $|ph\rangle$ follows provided $|ph\rangle$ is an inner product state since $|ph\rangle' = U|ph\rangle = |ph\rangle + Q|\ \rangle$. Notice also

that there are gauge transformations that change ψ but not the conditions (3.5) (or the opposite way around as will be shown below). For instance, the results in [18] suggests that we may have the representation (3.14) with $\psi = \mathcal{P}\Lambda(V)+\mathcal{P}\mu(\Pi+H(T))$ for arbitrary parameter μ.

Since the gauge theory we are considering does not contain any topological obstructions it is natural to expect (3.23) to represent all possible solutions for one definite sign in the exponential. However, this is *not* the case. It is impossible to get $-\psi$ by a unitary gauge transformation of ψ. Thus, there are two disconnected sets of solutions even in topologically trivial theories. Either there is a condition which excludes one of these sets or they are equivalent. In the next section we find that for cohomological quantum mechanics one set has negative norms and must be excluded. (This is a general property when we have an odd number of original constraints [18].)

The BRST invariant states (3.23) are defined up to zero norm states. In order to determine unique states representing the original theory we have to project out the BRST singlets [10], which we denote by $|s\rangle$. The BRST singlets have exactly the same form as (3.23) except that the $|\phi\rangle$-states satisfy two extra gauge fixing conditions. The allowed class of gauge fixing conditions was specified in [16]. For (3.14) and (3.21) the natural choice is [6]

$$\chi(T)|\phi\rangle = \bar{\mathcal{C}}|\phi\rangle = 0, \tag{3.24}$$

where $\chi(T)$ is a hermitian gauge fixing to $\Pi + H(T)$. ($\dot{\chi}(T) \neq 0$ is required.) The last condition in (3.24) completely ghost fixes $|\phi\rangle$ and makes the corresponding singlet $|s\rangle$ have ghost number zero. For (3.6) the natural gauge fixing conditions are

$$\Lambda(V)|\phi\rangle = \mathcal{P}|\phi\rangle = 0, \tag{3.25}$$

where $\Lambda'(V) \neq 0$ is required. In the following it will be shown that linear χ and Λ are always allowed and that a restriction to linear χ and Λ have no physical consequence.

All the solutions given here are formal. To actually determine true inner product solutions we must specify the representation of the original extended state space. When this is done we will be able to give wave function representations of the BRST singlets $|s\rangle$.

4 Wave function representations

The freedom in the choice of an extended state space in a BRST theory has to be investigated for each model under consideration. For cohomological quantum mechanics we assume the original theory to be regular and possible to quantize. It is therefore natural to assume the operators P_i, Q^i to span a Hilbert space, in which case P_i, Q^i must have real eigenvalues on appropriate eigenstates. However, time T and the Lagrange multiplier V as well as the ghosts belong to the unphysical part of the state space. This means that half of these variables must span an indefinite metric state space. The corresponding half of the hermitian operators have then imaginary eigenvalues on the appropriate eigenstates. Now exactly which half of the unphysical operators have these properties is not specified off-hand. The different choices for the fermions are simply related by phase factors like signs or factors of i, but this is not the case for the bosons. The ambiguity is vary large. T may *e.g.* have arbitrary complex eigenvalues [18]. Now even the separation into

physical and unphysical variables are not that simple either. In fact, physical operators should be BRST invariant, and P_i and Q^i are not. However, we know there exist BRST invariant operators representing the P_i- and Q^i-operators of the original theory. (These so called gauge invariant extended operators are expected to be closely related to P_i and Q^i and to have real eigenvalues (cf appendix A).) Below we let the possible wave function representations determine the possible choices of an extended state space.

Consider the particular BRST singlet

$$|s\rangle = e^{[Q,\psi]}|\phi\rangle, \quad \psi = \mathcal{P}\Lambda(V), \tag{4.1}$$

where $|\phi\rangle$ satisfies the conditions (3.5) and (3.24). We consider first the wave function representations of the $|\phi\rangle$-states. The conditions (3.5) and (3.24) ghost fix $|\phi\rangle$ to

$$|\phi\rangle = |\phi\rangle_B \otimes |0\rangle_{\mathcal{C}\bar{\mathcal{C}}}, \tag{4.2}$$

where $|0\rangle_{\mathcal{C}\bar{\mathcal{C}}}$ is a ghost vacuum, and where the bosonic part $|\phi\rangle_B$ satisfies

$$\Pi_V|\phi\rangle_B = 0, \quad \chi(T)|\phi\rangle_B = 0. \tag{4.3}$$

The formal wave function representation of the solutions is

$$\langle q^i, t, v, \mathcal{P}, \bar{\mathcal{P}}|\phi\rangle = \delta(\chi(t))\varphi(q,t), \tag{4.4}$$

where q^i, t, v, $\mathcal{P}$, $\bar{\mathcal{P}}$ are eigenvalues of the operators Q^i, T, V, $\mathcal{P}$, $\bar{\mathcal{P}}$ respectively. (One may of course also consider wave functions depending on the eigenvalues of the ghosts $\mathcal{C}, \bar{\mathcal{C}}$ instead of $\mathcal{P}, \bar{\mathcal{P}}$. However, in this case the right hand side would involve the delta function $\delta(\mathcal{C})\delta(\bar{\mathcal{C}})$ as an additional factor. We find the representation (4.4) more convenient to use here.) Now a solution of the type (4.4) is only consistent if $\chi(t)$ is real up to a constant phase factor which restricts the representation and/or $\chi(t)$. Furthermore, we must require $\chi(t) = 0$ to have a unique solution, $t = t_0$, which requires $\chi(t)$ to be a monotonic function of t. (Otherwise, we would have gauge inequivalent sectors and we would not be able to obtain the original theory.) This means that $\chi(t)$ always may be replaced by the linear expression $t - t_0$. Notice that the inner product of (4.4) leads to two zero factors from the $\mathcal{P}$ and $\bar{\mathcal{P}}$ integrations, and to two infinite factors from the v and t integrations. ($\dot{\chi}(t_0)$ is assumed to be a finite and φ is assumed to be such that $\int d^n q \varphi^* \varphi < \infty$.)

What is the wave function representation of the singlet (4.1)? The conditions (3.5) and (3.24) on $|\phi\rangle$ imply the following conditions on $|s\rangle$ using the gauge fixing fermion (4.1) (cf. (3.17) and the treatment in [6])

$$\begin{aligned}
&\Big(\Pi_V + i\Lambda'(V)\left(\Pi + H(T)\right) + \Lambda''(V)\bar{\mathcal{P}}\mathcal{P}\Big)|s\rangle = 0\\
&\Big(\mathcal{C} - i\Lambda'(V)\bar{\mathcal{P}}\Big)|s\rangle = \Big(\bar{\mathcal{C}} + i\Lambda'(V)\mathcal{P}\Big)|s\rangle = 0\\
&\chi(T - i\Lambda(V))|s\rangle = 0,
\end{aligned} \tag{4.5}$$

which yield the following solution in the wave function representation

$$\begin{aligned}
&\Psi(q,t,v,\mathcal{P},\bar{\mathcal{P}}) \equiv \langle q^i, t, v, \mathcal{P}, \bar{\mathcal{P}}|s\rangle =\\
&= e^{-i\Lambda'(v)\bar{\mathcal{P}}\mathcal{P}}\delta(\chi(t - i\Lambda(v)))e^{\Lambda(v)(-i\partial_t + H_S(t))}\varphi(q,t),
\end{aligned} \tag{4.6}$$

where $\varphi(q,t)$ is the same wave function as in (4.4). $H_S(t)$ is the Schrödinger representation of the operator $H(T)$ defined by

$$H_S(t)\langle q^i,t,v,\mathcal{P},\bar{\mathcal{P}}|\phi\rangle \equiv \langle q^i,t,v,\mathcal{P},\bar{\mathcal{P}}|H(T)|\phi\rangle. \tag{4.7}$$

The differential operator in (4.6) may be worked out explicitly (see (4.18) and (4.20) below).

The representations (4.4) and (4.6) are formal as long as we do not specify the properties of the eigenvalues. A precise choice of state space determines these eigenvalues. Since the argument of the delta function in (4.6) must be real up to a constant phase factor, we arrive at the following two natural options what regards the bosonic variables:

Case 1: t is real, v is imaginary, and $\Lambda(v)$ is imaginary

Case 2: t is imaginary, v is real, and $\chi(t)$ is imaginary

(One may also choose complex eigenvalues which correspond to representations between these two possibilities [18].) For the fermions we choose $\mathcal{P}$ real and $\bar{\mathcal{P}}$ imaginary ($= i\bar{\mathcal{P}}$, $\bar{\mathcal{P}}$ real). Some properties of the corresponding ghost states are given in appendix B.

Consider case 1 first. Put $v = iu$ where u is real and assume $\chi(t) = 0$ to have the unique solution $t = t_0$. The eigenstates of V have the properties [2, 22]

$$\begin{aligned} &V|iu\rangle = iu|iu\rangle, \quad \langle -iu| \equiv (|iu\rangle)^\dagger, \quad \langle iu|iu'\rangle = \delta(u-u') \\ &\int du|iu\rangle\langle iu| = \int du|-iu\rangle\langle -iu| = \mathbf{1}. \end{aligned} \tag{4.8}$$

We find then from (4.6) ($\mathcal{P}$, $\bar{\mathcal{P}}$ real)

$$\begin{aligned} \langle s|s\rangle &= \int d^nq dt du d\mathcal{P} d\bar{\mathcal{P}}\, \Psi^*(q,t,-iu,\mathcal{P},-i\bar{\mathcal{P}})\Psi(q,t,iu,\mathcal{P},i\bar{\mathcal{P}}) = \\ &= -\mathrm{sign}(\Lambda'(0)) \int d^nq \frac{1}{|\dot{\chi}(t_0)|^2}\varphi^*(q,t_0)\varphi(q,t_0) \end{aligned} \tag{4.9}$$

provided $i\Lambda(iu)$ is a real monotonic function. Since Λ is real for real argument, $\Lambda(iu)$ must be an odd function of u in order to be imaginary. Hence, if $i\Lambda(iu)$ is monotonic in u then $\Lambda(iu) = 0$ has the unique solution $u = 0$. (If $i\Lambda(iu)$ is not monotonic and $\Lambda(iu) = 0$ has several solutions then (4.9) is replaced by a sum of terms which is not what we should have.) Obviously a restriction to a linear $\Lambda(iu)$ has no physical consequence.

The expression (4.9) implies that $\langle s|s\rangle$ is only positive definite for $\Lambda' < 0$. (Notice that $\Lambda'(iu)$ is real.) This means that the two classes of gauge fixing characterized by opposite signs of ψ have opposite norms here. (This is in general the case if the number of original constraints is odd [18].) Thus, since we have positive norms for $\Lambda' < 0$ we have to exclude the choice $\Lambda' > 0$. The unitary gauge transformations can obviously not change the sign of Λ'.

Consider now case 2. For $t = iu$, $i\chi(iu)$ must be real and monotonic in u. Since χ is real for real argument, $i\chi(iu)$ must be odd in u. Thus, $\chi(iu) = 0$ has the unique solution $u = 0$. (A linear χ is always possible.) We find now with the same conventions as above

$$\begin{aligned} \langle s|s\rangle &= \int d^nq du dv d\mathcal{P} d\bar{\mathcal{P}}\, \Psi^*(q,-iu,v,\mathcal{P},-i\bar{\mathcal{P}})\Psi(q,iu,v,\mathcal{P},i\bar{\mathcal{P}}) = \\ &= -\mathrm{sign}(\Lambda'(v_0)) \int d^nq \frac{1}{|\dot{\chi}(0)|^2}\varphi^*(q,0)\varphi(q,0), \end{aligned} \tag{4.10}$$

provided $\Lambda(v)$ is monotonic such that $\Lambda(v)=0$ yields the unique value $v=v_0$. (If this is not the case we get a sum in (4.10) which we should not have.) Also (4.10) is positive definite for $\Lambda'<0$. This is a particular case of (4.9), since χ here is restricted and does not yield any explicit parameter t_0.

The only parameters the gauge fixing condition $\chi(T)|\phi\rangle=0$ introduces are t_0 and $\dot{\chi}$. Since the inner products of the BRST singlets $|s\rangle$ cannot depend on these gauge parameters the results above suggest that we should define the physical wave function by ($t_0=0$ in case 2)

$$\phi(q,t_0)\equiv\frac{1}{|\dot{\chi}(t_0)|}\varphi(q,t_0), \tag{4.11}$$

since (4.9) then yields

$$\langle s|s\rangle=\int d^nq\,\phi^*(q,t_0)\phi(q,t_0), \tag{4.12}$$

for $\Lambda'<0$. Obviously ϕ is independent of $\dot{\chi}$ since it is ϕ that is normalized by $|s\rangle$. The BRST quantization requires (4.12) to be independent of the gauge parameter t_0. It is therefore natural to expect that different values of t_0 may be reached by unitary gauge transformation of the form (3.9). An appropriate unitary gauge operator is

$$U(a)\equiv e^{i[Q,\rho]}=e^{-ia(\Pi+H(T))},\quad \rho\equiv-a\mathcal{P}, \tag{4.13}$$

where a is a real constant. This $U(a)$ does not alter the gauge fixing fermion ψ in (4.1) which means that

$$|s'\rangle=U(a)|s\rangle=e^{[Q,\psi]}|\phi'\rangle,\quad |\phi'\rangle=U(a)|\phi\rangle, \tag{4.14}$$

where $|\phi'\rangle$ satisfies the same conditions as $|\phi\rangle$ except for the gauge fixing condition which becomes

$$\chi(T-a)|\phi'\rangle=0, \tag{4.15}$$

since $U(a)\chi(T)U^{-1}(a)=\chi(T-a)$. Thus, $U(a)$ induces translations in t. We get therefore

$$\begin{aligned}&\langle q^i,t,v,\mathcal{P},\bar{\mathcal{P}}|\phi'\rangle=\delta(\chi(t-a))\varphi'(q,t)\equiv\delta(t-t_0-a)\phi'(q,t),\\&\langle q^i,t,v,\mathcal{P},\bar{\mathcal{P}}|\phi\rangle=\delta(\chi(t))\varphi(q,t)\equiv\delta(t-t_0)\phi(q,t),\end{aligned} \tag{4.16}$$

where

$$\phi'(q,t)=\frac{1}{|\dot{\chi}(t_0)|}\varphi'(q,t). \tag{4.17}$$

The relation $|\phi'\rangle=U(a)|\phi\rangle$ implies

$$\phi'(q,t_0+a)=e^{-a(\partial_t+iH_S(t))}\phi(q,t_0)\Big|_{t=t_0+a}. \tag{4.18}$$

This means that $\phi'(q,t)$ satisfies the Schrödinger equation with respect to t and that $\phi'(q,t_0)=\phi(q,t_0)$. The relation (4.14) implies furthermore that

$$\langle s|s\rangle=\langle s'|s'\rangle=\int d^nq\,\phi'^*(q,t)\phi'(q,t). \tag{4.19}$$

The wave function $\phi'(q,t)$ seems therefore to be a more appropriate physical wave function than $\phi(q,t_0)$ since it satisfies the Schrödinger equation. If we view $\phi'(q,t)$ as the physical wave function at arbitrary times then $\phi(q,t_0)$ is the physical wave function at $t=t_0$ since $\phi'(q,t_0)=\phi(q,t_0)$. However, a peculiar property of $\phi'(q,t)$ is that it depends on the gauge parameter t_0 on which no physical property should depend. On the other hand this is expected if we view $\phi'(q,t)$ as the gauge invariant extension of $\phi(q,t_0)$: That it satisfies the Schrödinger equation expresses its gauge invariance, and that $\phi'(q,t_0)=\phi(q,t_0)$ tells us that $\phi'(q,t)$ indeed is the gauge invariant extension of the wave function $\phi(q,t_0)$. (Compare the properties of the classical gauge invariant extensions considered in appendix A. There we have $-ad(H(t))$ instead of $iH_S(t)$.)

Eq.(4.18) may be rewritten as follows

$$\phi'(q,t)=U(t,t_0)\phi(q,t_0), \tag{4.20}$$

where

$$U(t,t_0)=\left\{\begin{array}{ll}\mathcal{T}\exp\{-i\int_{t_0}^{t}dt'H_S(t')\}, & t>t_0\\ \tilde{\mathcal{T}}\exp\{-i\int_{t_0}^{t}dt'H_S(t')\}, & t<t_0,\end{array}\right. \tag{4.21}$$

where $\mathcal{T}$ and $\tilde{\mathcal{T}}$ denote time and antitime ordering respectively. Since $U(t,t_0)$ is a unitary operator (4.20) confirms the relation (4.19). We notice that if $\phi(q,t_0)$ satisfies the Schrödinger equation, then $\phi'(q,t)$ is independent of t_0 due to (4.20) and we have $\phi'(q,t)=\phi(q,t)$. This could therefore to be a reasonable condition to impose.

In case 2 the parameter a in $U(a)$ (4.13) must be imaginary which implies that $U(a)$ no longer is unitary, and that $\chi(T-a)$ no longer is hermitian. However, this is consistent! (4.14) is still a singlet state. Instead of (4.18) we get that $\phi'(q,iu)$ satisfies the Schrödinger equation with imaginary time. However, since (4.15) implies

$$\langle\phi|\chi(T+a)=0 \tag{4.22}$$

we get

$$\langle s|s\rangle=\langle s|U^\dagger(a)U(a)|s\rangle=\int d^nq\,\phi'^*(q,0)\phi'(q,0)=\int d^nq\,\phi^*(q,0)\phi(q,0). \tag{4.23}$$

(The last equality is trivial since $\phi'(q,0)=\phi(q,0)$.) $\phi'(q,iu)$ for nonzero u does therefore not represent the singlets $|s\rangle$. An appropriate inner product for $\phi'(q,iu)$ would otherwise be $\int d^nq\,\phi'^*(q,-iu)\phi'(q,iu)$ which is positive definite only if $\phi'(q,iu)$ is even in u.

4.1 Physical wave functions as projections of BRST singlets

Since the singlet states represent the physical states, they should contain the wave functions of the original theory. We propose the following general projection from singlets to physical wave functions depending on the original coordinates and a parameter which possibly could represent time

> Impose boundary conditions determined by the conditions on $|\phi\rangle$ on the wave function representation Ψ of the BRST singlets $|s\rangle$. (A)

Symbolically this may be stated as follows

$$\phi_{phys}(q,t_0) = \Psi|_{b_i=c_i=0}\,, \tag{4.24}$$

where b_i and c_i are eigenvalues or Weyl symbols of the operators B_i and C_i in the defining conditions $B_i|\phi\rangle = C_i|\phi\rangle = 0$. Here we have $B_i = \{\Pi_V, \mathcal{C}\}$ and $C_i = \{\bar{\mathcal{C}}, \chi(T)\}$ which in case 1 yields the boundary conditions $\pi_u = \mathcal{C} = 0$ and $\bar{\mathcal{C}} = \chi(t) = 0$. Hence, we find from (4.24) and (4.6) ($\mathcal{P}$ and $\bar{\mathcal{P}}$ are real and $\lambda(u) = i\Lambda(iu)$ is a real monotonic function of u. $\lambda'(u) = -\Lambda' > 0$)

$$\phi_{phys}(q,t_0) \equiv \int dud\mathcal{P}d\bar{\mathcal{P}}\,\Psi(q,t_0,iu,\mathcal{P},i\bar{\mathcal{P}}) = \phi(q,t_0), \tag{4.25}$$

i.e. exactly (4.11) which we derived in another way above. Notice that $\chi(t) = 0$ is equivalent to $t - t_0 = 0$, and that $\mathcal{C} = \bar{\mathcal{C}} = \pi_u = 0$ is equivalent to an integration over $\mathcal{P}, \bar{\mathcal{P}}$ and u due to Fourier transformations. In case 2 we get similarly $\phi(q,0)$ as suggested by (4.10).

There is also an interesting partial projection which may be stated as follows:

> Impose the conditions on $|\phi\rangle$ except for the gauge fixing conditions as boundary conditions on the wave function representation of the BRST singlets $|s\rangle$. (B)

The C_i operators contain both ghost fixings and gauge fixings of the original invariances. The prescription B requires us not to impose the last ones as boundary conditions on Ψ. In case 1 the prescription B requires us here to put $\mathcal{C}, \bar{\mathcal{C}}$ and π_u equal to zero but not $t = t_0$. We get therefore from (4.6).

$$\bar{\phi}(q,t,t_0) \equiv \int dud\mathcal{P}d\bar{\mathcal{P}}\,\Psi(q,t,iu,\mathcal{P},i\bar{\mathcal{P}}) = e^{-\lambda(\partial_t + iH_S(t))}\phi(q,t)\Big|_{\lambda=t-t_0}\,, \tag{4.26}$$

which means that

$$\bar{\phi}(q,t,t_0) = \phi'(q,t). \tag{4.27}$$

Thus, $\bar{\phi}(q,t,t_0)$ is the appropriate physical wave function which satisfies the Schrödinger equation.

In case 2 the partially projected wave function is given by ($\Lambda' < 0$)

$$\bar{\phi}(q,iu) \equiv \int dvd\mathcal{P}d\bar{\mathcal{P}}\,\Psi(q,iu,v,\mathcal{P},i\bar{\mathcal{P}}) = \frac{1}{|\dot{\chi}(0)|}e^{-\lambda(\partial_u - H_S(iu))}\varphi(q,0)\Big|_{\lambda=u}, \tag{4.28}$$

which satisfies the Schrödinger equation for imaginary time, iu. Obviously $\bar{\phi}(q,iu) = \phi'(q,iu)$. If we also impose the gauge fixing condition $u = 0$ we get $\bar{\phi}(q,0) = \phi'(q,0) = \phi(q,0)$.

It is remarkable that the physical wave function which follows from the projection A above also is directly related to the wave function representation of the $|\phi\rangle$-state according to (4.4). We have

$$\langle q^i, t, v, \mathcal{P}, \bar{\mathcal{P}}|\phi\rangle = \delta(t-t_0)\phi(q,t). \tag{4.29}$$

From this result we propose the following general projection prescription:

> The physical wave functions may be obtained from the wave function of the $|\phi\rangle$-state by imposing the dual conditions to the gauge fixing conditions on $|\phi\rangle$ as boundary conditions. (C)

By a dual condition we mean the canonical conjugate condition or more precisely the corresponding gauge invariance. In case 1 the dual to the gauge fixing condition, $\chi = 0$, is as boundary condition $\pi + H = 0$. This condition may roughly be replaced by an integration over time t which obviously leads to the physical wave function. Notice that we can only obtain a gauge fixed physical wave function from the $|\phi\rangle$-state. (The total set of boundary conditions satisfied by $\langle q^i, t, v, \mathcal{P}, \bar{\mathcal{P}}|\phi\rangle$ corresponds to the dual set (3.6) and (3.25) of allowed conditions on $|\phi\rangle$, which also will be considered in the next section.)

It is only projection B that leads to physical wave functions satisfying the basic gauge invariance which here is the Schrödinger equation.

4.2 Projected physical propagators

Consider the gauge transformed singlet

$$|\tilde{s}\rangle = U(\alpha)|s\rangle, \quad U(\alpha) = e^{i\alpha[Q,\rho]}, \quad \rho \equiv \mathcal{P}(T - t_0), \tag{4.30}$$

where α is a real parameter. With $|s\rangle$ given by (4.1) we find

$$|\tilde{s}\rangle = e^{\frac{\gamma}{2}[Q,\psi]}|\phi\rangle, \tag{4.31}$$

where γ is a positive constant which depends on α in (4.30). Note that $U(\alpha)|\phi\rangle \equiv |\phi\rangle$. We have therefore

$$\langle s|s\rangle = \langle\tilde{s}|\tilde{s}\rangle = \langle\tilde{s}|U^{\dagger}(b)U(a)|\tilde{s}\rangle = \langle\phi''|e^{\gamma[Q,\psi]}|\phi'\rangle, \tag{4.32}$$

where $U(a)$ and $U(b)$ are given by (4.13) and where $|\phi'\rangle$ and $|\phi''\rangle$ are defined by (4.14). This inner product may be calculated in the representation 1 where t is real and v is imaginary. We find then

$$\langle s|s\rangle = \langle\phi''|e^{\gamma[Q,\psi]}|\phi'\rangle = \int d^{n+4}R''d^{n+4}R'\langle\phi|R''^{*}\rangle\langle R''|e^{\gamma[Q,\psi]}|R'^{*}\rangle\langle R'|\phi'\rangle, \tag{4.33}$$

where $R \equiv \{q^i, t, iu, \mathcal{P}, i\bar{\mathcal{P}}\}$. From (4.4) we have

$$\langle R''^{*}|\phi''\rangle = \delta(\chi(t''))\varphi(q'', t''), \quad \langle R'|\phi'\rangle = \delta(\chi(t' - a))\varphi'(q', t'). \tag{4.34}$$

Hence, we get

$$\langle s|s\rangle = \langle\phi''|e^{\gamma[Q,\psi]}|\phi'\rangle = \int d^nq''d^nq'\phi''^{*}(q'', t_0 + b)K(q'', q'; t_0 + b, t_0 + a)\phi'(q', t_0 + a), \tag{4.35}$$

where

$$K(q'', q'; t'', t') \equiv \int du''d\mathcal{P}''d\bar{\mathcal{P}}''du'd\mathcal{P}'d\bar{\mathcal{P}}'\langle q''^i, t'', iu'', \mathcal{P}'', i\bar{\mathcal{P}}''|e^{\gamma[Q,\psi]}|q'^i, t', iu', \mathcal{P}', i\bar{\mathcal{P}}'\rangle. \tag{4.36}$$

Since (4.35) also may be written as

$$\begin{aligned}\langle s|s\rangle &= \langle\phi''|e^{\gamma[Q,\psi]}|\phi'\rangle = \\ &= \int d^nq''d^nq'\bar{\phi}^*(q'',t_0+b,t_0)K(q'',q';t_0+b,t_0+a)\bar{\phi}(q',t_0+a,t_0),\end{aligned} \tag{4.37}$$

where $\bar{\phi}(q',t,t_0)$ is the physical wave function, K in (4.36) must be the physical propagator. Note that the expression (4.36) involves exactly the same integrations as in the formula for the partially projected physical wave function $\bar{\phi}(q',t,t_0)$ given in (4.26). Thus, we have arrived at the following general prescription for physical propagators:

> Impose as boundary conditions on the extended propagator $\langle R''|e^{\gamma[Q,\psi]}|R'^*\rangle$ the conditions on $|\phi\rangle$ except for the gauge fixing conditions. (D)

(In principle we may here also impose the gauge fixing conditions, but then they should be different on the left and right-hand side which effectively is what the $|\phi\rangle$-states (4.34) do in (4.35).) The same prescription is also valid for the standard BFV path integral propagators as will be shown in section 6.

We may explicitly check that (4.36) is the physical propagator by performing the integrations. We find

$$\begin{aligned}&K(q,q';t,t') = -\gamma\int_{-\infty}^{\infty}du\Lambda'(iu)e^{\gamma\Lambda(iu)(-i\partial_t+H_S(t))}\delta^n(q-q')\delta(t-t') = \\ &= \varepsilon\int_{-\infty}^{\infty}d\lambda e^{-\varepsilon\lambda(\partial_t+iH_S(t))}\delta^n(q-q')\delta(t-t') = \varepsilon\langle q^i,t|q'^i,t'\rangle, \\ &\langle q^i,t|q'^i,t'\rangle \equiv e^{-\lambda(\partial_t+iH_S(t))}\delta^n(q-q')\Big|_{\lambda=t-t'}, \quad \varepsilon=-\text{sign}(\gamma\Lambda').\end{aligned} \tag{4.38}$$

The change of variable $u\to\lambda$ in the second equality is allowed since $\lambda(u)\equiv\pm\gamma i\Lambda(iu)$ is a real monotonic function of u. The sign is chosen such that $\lambda'(u)>0$. Eq.(4.38) is the correct physical propagator for positive norm states when $\Lambda'<0$ since $\gamma>0$. (For $\gamma<0$ $\Lambda'>0$ yields the correct physical propagator.) Its gauge invariance follows from the last equality in (4.38). We have

$$\langle q^i,t|(\Pi+H(T))|q'^i,t'\rangle = (-i\partial_t+H_S(t))\langle q^i,t|q'^i,t'\rangle = 0. \tag{4.39}$$

Propagators may be expressed in terms of path integrals (see section 6), and that path integrals could satisfy the original constraints was first proposed for the wave function of the universe in [24]. In [25] (see also [2]) this was shown to be a general feature of BFV path integrals if one imposes the generalization of the boundary conditions considered in this section (given in [21]). These boundary conditions are the simplest allowed conditions one may impose and seems also special for these properties. However, our results of section 6 suggest that these properties should also be the valid for all allowed boundary conditions provided an appropriate gauge fixing fermion is chosen.

That the expression (4.35) is independent of a and b follows since

$$\bar{\phi}(q,t,t_0) = \int d^nq'^i\langle q^i,t|q'^i,t'\rangle\bar{\phi}(q',t',t_0). \tag{4.40}$$

In case 2 we cannot derive the physical propagator along the lines followed in case 1 since the singlets are not represented by $\phi'(q,iu)$ for nonzero u. However, we may still

define the physical propagator by the prescription D. We find then

$$K(q,q';iu,iu')\equiv\int dvd\mathcal{P}d\bar{\mathcal{P}}dv'd\mathcal{P}'d\bar{\mathcal{P}}'\langle q^i,iu,v,\mathcal{P},i\bar{\mathcal{P}}|e^{\gamma[Q,\psi]}|q'^i,iu',v',\mathcal{P}',i\bar{\mathcal{P}}'\rangle=$$
$$=\varepsilon\int_{-\infty}^{\infty}d\lambda e^{-\lambda(\partial_u-H_S(iu))}\delta^n(q-q')\delta(u-u')=\varepsilon\langle q^i,iu|q'^i,iu'\rangle,$$
$$\langle q^i,iu|q'^i,iu'\rangle=e^{-\lambda(\partial_u-H_S(iu))}\delta^n(q-q')\Big|_{\lambda=u'-u},\varepsilon\equiv-\mathrm{sign}(\gamma\Lambda'),\tag{4.41}$$

which indeed is the appropriate propagator for imaginary time and $\gamma\Lambda'<0$.

These projected propagators will also be considered within the path integral formulation in section 6.

5 Representations of other BRST singlets

So far we have considered the particular BRST singlet (3.14). All other BRST singlets $|\tilde{s}\rangle$ are obtained from (3.14) by means of extended unitary gauge transformations,

$$|\tilde{s}\rangle=U|s\rangle,\quad U=\exp\{i[Q,\rho]\},\tag{5.1}$$

where ρ is an odd, hermitian operator with ghost number one. This implies

$$|\tilde{s}\rangle=e^{[Q,\tilde{\psi}]}|\tilde{\phi}\rangle,\tag{5.2}$$

where $\tilde{\psi}=\tilde{\mathcal{P}}\Lambda(\tilde{V})$, and where $|\tilde{\phi}\rangle$ satisfies the conditions

$$\tilde{\Pi}_V|\phi\rangle=\tilde{\mathcal{C}}|\phi\rangle=0,$$
$$\chi(\tilde{T})|\phi\rangle=\tilde{\bar{\mathcal{C}}}|\phi\rangle=0.\tag{5.3}$$

Note that all tilde operators are defined by (3.10), *i.e.* $\tilde{A}\equiv UAU^{\dagger}$, where U is an extended gauge transformation of the form (5.1). The two different sets in (3.23) are contained in (5.2) as long as we do not specify $\Lambda(\tilde{V})$. (We have either $\Lambda'(\tilde{V})>0$ or $\Lambda'(\tilde{V})<0$.) Since the BRST charge (3.3) may be written as

$$Q=\tilde{\mathcal{C}}(\tilde{\Pi}+\tilde{H}(\tilde{T}))+\tilde{\bar{\mathcal{P}}}\tilde{\Pi}_V,\tag{5.4}$$

where $\tilde{H}=UHU^{\dagger}$, we may view the tilde operators as the basic ones. This shows that we should get exactly the same results as above if we make use of tilde operators. However, in terms of the original variables the situation is different. Since $|\tilde{s}\rangle$ is different from $|s\rangle$ their wave function representations should be different. Below we treat therefore another BRST singlet in detail.

We consider here the particular BRST singlets

$$|s\rangle=e^{[Q,\psi]}|\phi\rangle,\quad\psi=\bar{\mathcal{C}}\chi(T)\tag{5.5}$$

where $|\phi\rangle$ satisfies the conditions (3.6) and (3.25), *i.e.*

$$\bar{\mathcal{P}}|\phi\rangle=(\Pi+H(T))\,|\phi\rangle=0,$$
$$\Lambda(V)|\phi\rangle=\mathcal{P}|\phi\rangle=0,\tag{5.6}$$

which are dual to (3.5) and (3.24). These singlets are contained in the singlets (5.2). Take *e.g.* the gauge transformation (3.12) with $\theta = \pi/2,\ 3\pi/2$. The wave function representation of the $|\phi\rangle$-state satisfying (5.6) is

$$\langle q^i, t, v, \mathcal{C}, \bar{\mathcal{C}}|\phi\rangle = \delta(\Lambda(v))\varphi(q, t, v), \tag{5.7}$$

where $\varphi(q, t, v)$ satisfies the Schrödinger equation with respect to t. This is quite different from (4.4) which did not involve any Schrödinger equation. Again $\Lambda(v) = 0$ is required to yield the unique solution $v = v_0$ which implies that we always may choose $\Lambda(v)$ to be linear in v. The conditions (5.6) imply the following conditions on $|s\rangle$ in (5.5):

$$\begin{aligned}
&\Big(\Pi + H(T) + i\dot{\chi}(T)\Pi_V + \ddot{\chi}(T)\mathcal{C}\bar{\mathcal{C}}\Big)|s\rangle = 0\\
&\Big(\mathcal{P} + i\dot{\chi}(T)\bar{\mathcal{C}}\Big)|s\rangle = \Big(\bar{\mathcal{P}} - i\dot{\chi}(T)\mathcal{C}\Big)|s\rangle = 0\\
&\Lambda(V - i\chi(T))|s\rangle = 0,
\end{aligned} \tag{5.8}$$

which yield the following solution in the wave function representation

$$\begin{aligned}
&\Psi(q, t, v, \mathcal{C}, \bar{\mathcal{C}}) \equiv \langle q^i, t, v, \mathcal{C}, \bar{\mathcal{C}}|s\rangle =\\
&= e^{i\dot{\chi}(t)\bar{\mathcal{C}}\mathcal{C}}\delta(\Lambda(v - i\chi(t))\varphi(q, t, v - i\chi(t)),
\end{aligned} \tag{5.9}$$

where $\varphi(q, t, v)$ is the same wave function as in (5.7).

Let us calculate the inner product for the following two natural representations:

Case a: t imaginary, v real, and $\chi(t)$ imaginary. $\mathcal{C}$ real and $\bar{\mathcal{C}}$ imaginary ($= -i\bar{\mathcal{C}}$, $\bar{\mathcal{C}}$ real).

Case b: t real, v imaginary, and $\Lambda(v)$ imaginary. $\mathcal{C}$ real and $\bar{\mathcal{C}}$ imaginary ($= -i\bar{\mathcal{C}}$, $\bar{\mathcal{C}}$ real).

(By means of the gauge transformations (3.12) with $\theta = \pi/2,\ 3\pi/2$ they correspond to the two cases considered in section 4.)

Following the treatment in section 4 we find here in case a,

$$\begin{aligned}
\langle s|s\rangle = &\int du dv d^n q d\bar{\mathcal{C}} d\mathcal{C}\, \Psi^*(q, -iu, v, \mathcal{C}, i\bar{\mathcal{C}})\, \Psi(q, iu, v, \mathcal{C}, -i\bar{\mathcal{C}}) =\\
= \mathrm{sign}(\dot{\chi}(0)) &\int d^n q \frac{1}{|\Lambda'(v_0)|^2}\varphi^*(q, 0, v_0)\varphi(q, 0, v_0),
\end{aligned} \tag{5.10}$$

provided $i\chi(iu)$ and $\Lambda(v)$ are monotonic functions. In case b we find,

$$\begin{aligned}
\langle s|s\rangle = &\int dt du d^n q d\bar{\mathcal{C}} d\mathcal{C}\, \Psi^*(q, t, -iu, \mathcal{C}, i\bar{\mathcal{C}})\Psi(q, t, iu, \mathcal{C}, -i\bar{\mathcal{C}}) =\\
= \mathrm{sign}(\dot{\chi}(t_0)) &\int d^n q \frac{1}{|\Lambda'(0)|^2}\varphi^*(q, t_0, 0)\varphi(q, t_0, 0),
\end{aligned} \tag{5.11}$$

provided $\chi(t)$ and $i\Lambda(iu)$ are monotonic functions. Positive norms requires $\dot{\chi} > 0$ in both cases. Since $\varphi(q, t_0, 0)$ satisfies the Schrödinger equation with respect to t_0 we get in case b

$$\langle s|s\rangle = \mathrm{sign}(\dot{\chi}(t_0)) \int d^n q \frac{1}{|\Lambda'(0)|^2}\varphi^*(q, 0, 0)\varphi(q, 0, 0). \tag{5.12}$$

We may now compare (5.10) and (5.12) with (4.9) and (4.10). We find then agreement if we replace $\Lambda(v)$ by $\chi(-t)$ or $-\chi(t)$ as required by the gauge transformation (3.12) with $\theta = \pi/2,\ 3\pi/2$. We should also exchange v_0 and t_0.

From (5.10) we may identify a physical wave function in case a to be given by ($\dot{\chi} > 0$)

$$\phi_{(a)}(q, v_0) \equiv \frac{1}{|\Lambda'(v_0)|}\varphi(q, 0, v_0), \tag{5.13}$$

and in case b ($\dot{\chi} > 0$)

$$\phi_{(b)}(q, t_0) \equiv \frac{1}{|\Lambda'(0)|}\varphi(q, t_0, 0). \tag{5.14}$$

$\phi_{(a)}$ does not satisfy the Schrödinger equation with respect to v_0 as in case 1 in the previous section with t_0 replaced by v_0. On the other hand, the expression (5.14), which has no counterpart in the previous section, does satisfy the Schrödinger equation. Notice that $\langle s|s\rangle = \int d^n q \phi^*_{(b)}(q, t_0)\phi_{(b)}(q, t_0)$ for $\dot{\chi}(t) > 0$. The representation with real time and the Schrödinger equation as boundary condition is particular. One may notice that translations in v is performed by the unitary gauge operator

$$U(b) \equiv e^{i[Q,\rho]} = e^{-ib\Pi_V}, \quad \rho = -b\bar{\mathcal{C}}. \tag{5.15}$$

We have

$$\Lambda(v - b)|\phi'\rangle = 0, \quad |\phi'\rangle = U(b)|\phi\rangle, \tag{5.16}$$

which implies

$$\begin{aligned} \langle q^i, t, v, \mathcal{C}, \bar{\mathcal{C}}|\phi'\rangle &= \delta(\Lambda(v - b))\varphi'(q, t, v) = \\ &= \delta(v - b - v_0)\frac{1}{|\Lambda'(0)|}\varphi'(q, t, v_0 + b), \end{aligned} \tag{5.17}$$

and

$$\varphi'(q, t, v_0 + b) = \varphi(q, t, v_0). \tag{5.18}$$

Thus, no Schrödinger equation is derived here in case a.

We may also derive projected wave functions from the general rules in section 4. Assuming $\dot{\chi} > 0$ we find in case a for the partial projection B (without imposing $\Lambda = 0$)

$$\bar{\phi}(q, v, v_0) \equiv \int du d\bar{\mathcal{C}} d\mathcal{C}\, \Psi(q, iu, v, \mathcal{C}, -i\bar{\mathcal{C}}) = \frac{1}{|\Lambda'(v_0)|}\varphi(q, if(v - v_0), v_0). \tag{5.19}$$

(To set $\mathcal{P}, \bar{\mathcal{P}}$ to zero and $\pi = -H$ at $t = 0$ is equivalent to an integration over $\mathcal{C}, \bar{\mathcal{C}}$ and $t = iu$.) The function $f(v) = -i\chi^{-1}(-iv)$ is real and satisfies $f(0) = 0$. Notice that $\bar{\phi}(q, v, v_0)$ satisfies a Schrödinger equation with respect to v. In case b we find instead the partially projected wave function

$$\bar{\phi}(q, u, t_0) \equiv \int dt d\bar{\mathcal{C}} d\mathcal{C}\, \Psi(q, t, iu, \mathcal{C}, -i\bar{\mathcal{C}}) = \frac{1}{|\Lambda'(0)|}\varphi(q, t_0 + f(u), 0), \tag{5.20}$$

which satisfies a Schrödinger equation with respect to u. The function $f(u)$ is here given by $f(u) = \chi^{-1}(u) - t_0$ and is real satisfying $f(0) = 0$.

Effectively the physical projected wave functions (5.19) and (5.20) satisfy the Schrödinger equation for imaginary and real times respectively which are exactly the reality properties of time chosen in the two representations! If we also impose the gauge fixing conditions $\Lambda = 0$ we find the physical wave functions (5.13) and (5.14). As in the previous section only projected wave functions with real time represent the BRST singlets in the projection B. In case b we have obviously ($\dot{\chi} > 0$)

$$\langle s|s\rangle = \int d^n q \bar{\phi}^*(q, u, t_0)\bar{\phi}(q, u, t_0), \tag{5.21}$$

from (5.11). However, there is no such relation in case a for $\bar{\phi}(q, v, v_0)$ in (5.19).

The physical wave functions (5.13) and (5.14) are also here directly related to the wave function representations of the $|\phi\rangle$-states. We have in case a

$$\langle q^i, iu, v, \mathcal{C}, -i\bar{\mathcal{C}}|\phi\rangle = \delta(v - v_0)\frac{1}{|\Lambda'(v_0)|}\varphi(q, iu, v_0), \tag{5.22}$$

and in case b

$$\langle q^i, t, iu, \mathcal{C}, -i\bar{\mathcal{C}}|\phi\rangle = \delta(u)\frac{1}{|\Lambda'(0)|}\varphi(q, t, 0). \tag{5.23}$$

The projection prescription for these wave functions given in subsection 4.2 requires us to impose the dual boundary condition to the gauge fixing condition $\Lambda = 0$ which is $\pi_v = 0$ in case a. This is equivalent to an integration over v which yields $\frac{1}{|\Lambda'(v_0)|}\varphi(q, iu, v_0)$ which is equal to (5.13) for $u = 0$, and which also only represents singlets at $u = 0$. In case b we should impose $\pi_u = 0$ which is equivalent to an integration over u. We get then $\frac{1}{|\Lambda'(0)|}\varphi(q, t_0, 0)$,*i.e.* exactly (5.14).

Consider the following gauge transformed singlet (5.5)

$$\begin{aligned} &|\tilde{s}\rangle = U(\beta)|s\rangle = e^{\frac{\gamma}{2}[Q,\psi]}|\phi\rangle, \\ &U(\beta) = e^{i\beta[Q,\rho]}, \quad \rho \equiv \bar{\mathcal{C}}(V - v_0), \end{aligned} \tag{5.24}$$

where γ is a positive constant which depends on the real parameter β. We have then

$$\begin{aligned} &\langle s|s\rangle = \langle\tilde{s}|\tilde{s}\rangle = \langle\tilde{s}|U(b)|\tilde{s}\rangle = \langle\phi|e^{\gamma[Q,\psi]}|\phi'\rangle = \\ &= \int d^{n+4}R d^{n+4}R' \langle\phi|R^*\rangle\langle R|e^{\gamma[Q,\psi]}|R'^*\rangle\langle R'|\phi'\rangle, \end{aligned} \tag{5.25}$$

where $R \equiv \{q^i, t, v, \mathcal{C}, \bar{\mathcal{C}}\}$. From (5.6) and (5.16) we have

$$\langle R^*|\phi\rangle = \delta(\Lambda(v^*))\varphi(q, t^*, v^*), \quad \langle R'|\phi'\rangle = \delta(\Lambda(v' - b))\varphi'(q', t', v'), \tag{5.26}$$

where in distinction to (4.34) φ depends on both t and v. In case a we get therefore ($t = iu$)

$$\langle\phi|e^{\gamma[Q,\psi]}|\phi'\rangle = \int d^n q d^n q' du du' \phi^*(q, -iu, v_0)A(q, q'; iu, iu'; v_0, v_0 + b)\phi'(q', iu', v_0 + b), \tag{5.27}$$

where

$$\begin{aligned} &A(q, q'; iu, iu'; v, v') \equiv \int d\bar{\mathcal{C}}d\mathcal{C}d\bar{\mathcal{C}}'d\mathcal{C}'\langle q^i, iu, v, \mathcal{C}, -i\bar{\mathcal{C}}|e^{\gamma[Q,\psi]}|q'^i, iu', v', \mathcal{C}', -i\bar{\mathcal{C}}'\rangle = \\ &= \gamma\dot{\chi}(iu)\delta(u - u')\delta(v - v' - i\gamma\chi(iu))\delta^n(q - q'). \end{aligned} \tag{5.28}$$

This inserted into (5.27) yields (5.10).

In case b t is real,v imaginary ($= iu$) as well as b in (5.13) (($= id$)). We get

$$\langle\phi|e^{\gamma[Q,\psi]}|\phi'\rangle = \int d^nqd^nq'dtdt'\phi^*(q,t,0)A(q,q';t,t';0,id)\phi'(q',t',id), \quad (5.29)$$

where

$$A(q,q';t,t';iu,iu') \equiv \int d\bar{\mathcal{C}}d\mathcal{C}d\mathcal{C}'d\bar{\mathcal{C}}'\langle q^i,t,iu,\mathcal{C},-i\bar{\mathcal{C}}|e^{\gamma[Q,\psi]}|q'^i,t',iu',\mathcal{C}',-i\bar{\mathcal{C}}'\rangle =$$
$$= \gamma\dot{\chi}(t)\delta(t-t')\delta(u-u'-\gamma\chi(t))\delta^n(q-q'). \quad (5.30)$$

This inserted into (5.29) yields (5.11).

The physical propagators should according to the rule in subsection 4.2 be given by

$$K(q,q';v,v') \equiv \int dudu'A(q,q';iu,iu';v,v') = \text{sign}(\gamma\dot{\chi})\delta^n(q-q') \quad (5.31)$$

in case a and

$$K(q,q';iu,iu') \equiv \int dtdt'A(q,q';t,t';iu,iu') = \text{sign}(\gamma\dot{\chi})\delta^n(q-q') \quad (5.32)$$

in case b. Thus, although the above results are perfectly consistent, (5.31) and (5.32) are not the correct propagators for the Schrödinger equation. Obviously v, v' in (5.31) and u, u' in (5.32) are not time parameters. K does not depend on them. In fact, (5.31) and (5.32) may be interpreted as the physical propagators in the limit of equal times. (Notice the delta-functions in (5.28) and (5.30) and that $\phi'(q',iu',v_0+b) = \phi(q',iu',v_0)$ in (5.27) and $\phi'(q',t',id) = \phi(q',t',0)$ in (5.29) due to (5.18).) To get the correct propagators for different times we must use another gauge fixing fermion. In fact, to make sure always to obtain appropriate propagators we must use a gauge fixing fermion which is valid for any choices of $|\phi\rangle$-states. This is also what is required in the BFV path integral formulation. This will be demonstrated for the above case within the path integral formulation below.

6 Path integrals for propagators

In ordinary quantum mechanics we have the path integral representation

$$\langle q'',t''|q',t'\rangle = \langle q'',t'|U(t'',t')|q',t'\rangle =$$
$$= \int_{Path}\prod_t \frac{d^nqd^np}{(2\pi)^n}\exp\Big\{i\int_{t'}^{t''}dt\Big(p\cdot\dot{q} - H(p,q,t)\Big)\Big\} \quad (6.1)$$

where $U(t'',t')$ is given by (4.21) with the Schrödinger Hamiltonian $H_S(t)$ replaced by the operator $H(P,Q,t)$. $H(p,q,t)$ is the Weyl symbol of $H(P,Q,t)$ defined by

$$H(p,q,t) = \frac{1}{(2\pi)^n}\int d^nud^nvH(u,v,t)e^{-i(q\cdot u+p\cdot v)}$$
$$H(P,Q,t) = \frac{1}{(2\pi)^n}\int d^nud^nvH(u,v,t)e^{-i(Q\cdot u+P\cdot v)}. \quad (6.2)$$

The equality (6.1) may be obtained by means of the time slice formula. It is strictly valid for bosonic coordinates in which case (6.1) is equivalent to (4.38). (For each fermionic

coordinate one has to remove a factor 2π in (6.1),(6.2) and insert an extra factor i in the exponentials (only in the first term in (6.1)).)

A similar procedure may be used to obtain path integral representations for the inner products of the BRST singlets. Consider again the inner product (4.32). According to (4.33) we have then

$$\langle s|s\rangle = \langle\phi|e^{\gamma[Q,\psi]}|\phi'\rangle = \int d^{n+4}R''d^{n+4}R'\langle\phi'|R''^{*}\rangle\langle R''|e^{\gamma[Q,\psi]}|R'^{*}\rangle\langle R'|\phi\rangle, \quad (6.3)$$

where $R' \equiv \{t', v', \mathcal{P}', \bar{\mathcal{P}}', q'^{i}\}$ etc. If we set $\gamma \equiv \tau'' - \tau'$ then we obtain in analogy to (6.1) the formal path integral representation

$$\langle R''|e^{\gamma[Q,\psi]}|R'^{*}\rangle \equiv \langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^{*}\rangle =$$
$$= \int_{Path} \prod_{\tau} \frac{d^{n+4}Rd^{n+4}P}{(2\pi)^{n+2}} \exp\Big\{i\int_{\tau'}^{\tau''} d\tau\Big(P\cdot\dot{R} - i[Q,\psi]_W\Big)\Big\}, \quad (6.4)$$

where $R(\tau') \equiv R'$ and $R(\tau'') \equiv R''$, and where $[Q,\psi]_W$ is the Weyl symbol of $[Q,\psi]$ defined in (6.2) and in appendix B for fermions. Notice that the τ-parameter is introduced in a completely ad hoc manner. Since $[Q,\psi]$ has no explicit τ-dependence it may be viewed as a conserved Hamiltonian with respect to τ. Furthermore, since the left-hand side is independent of γ the right-hand side is independent of τ. *Thus, reparametrization invariance is implied.* (It is the unitary gauge transformation (4.30) which is behind this.) As a general statement of expressions like (6.4) the following may be stated: The reality properties of the effective action depends on the chosen representation. Typically it may be chosen either to be real or imaginary [20]. The expression (6.4) is what is prescribed by the BFV formalism if the effective action is real and if $[Q,\psi]_W$ is the Poisson bracket of the BRST charge and the gauge fixing factor. This is true in an appropriate representation if Q and ψ are sufficiently simple. This is the case here (see subsection 6.2 below).

Let us calculate (6.4) explicitly. We consider then the representation in which t is real and v and π_v are imaginary ($v = iu$, $\pi_v = -i\pi_u$ where u, π_u are real). Furthermore, we choose $\mathcal{C}, \mathcal{P}$ real and $\bar{\mathcal{C}}, \bar{\mathcal{P}}$ imaginary ($= -i\bar{\mathcal{C}}, i\bar{\mathcal{P}}$; $\bar{\mathcal{C}}, \bar{\mathcal{P}}$ real). For the gauge fixing fermion (4.1) we find

$$\langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^{*}\rangle = \int_{Path}\prod_{\tau}\frac{d^n q dt du d\bar{\mathcal{C}} d\mathcal{C} d^n p d\pi d\pi_u d\mathcal{P} d\bar{\mathcal{P}}}{(2\pi)^{n+2}} \times$$
$$\exp\Big\{i\int_{\tau'}^{\tau''} d\tau\Big(p_i\dot{q}^i + \pi\dot{t} + \pi_u\dot{u} + i\mathcal{C}\dot{\mathcal{P}} + i\bar{\mathcal{C}}\dot{\bar{\mathcal{P}}} - \lambda(\pi + H(t)) + i\lambda'\bar{\mathcal{P}}\mathcal{P}\Big)\Big\}, \quad (6.5)$$

where $\lambda(u) \equiv i\Lambda(iu)$ and $\lambda'(u) = -\Lambda'(iu)$. Notice that the effective Lagrangian is real in the chosen representation. (Formally we may choose rather arbitrary complex eigenvalues of the ghosts. However, strictly this is not true when we use the time slice formulation since there are not the same number of slices in $\mathcal{C}, \bar{\mathcal{C}}$ as in $\mathcal{P}, \bar{\mathcal{P}}$.) Integrations over the ghosts $\mathcal{C}$, $\bar{\mathcal{C}}$, and π, π_u, yields

$$\langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^{*}\rangle = \int_{Path}\prod_{\tau}\frac{d^n q dt du d^n p d\mathcal{P} d\bar{\mathcal{P}}}{(2\pi)^n}\delta(\dot{u})\delta(\lambda - \dot{t})\delta(\dot{\bar{\mathcal{P}}})\delta(\dot{\mathcal{P}}) \times$$
$$\exp\Big\{i\int_{\tau'}^{\tau''} d\tau\Big(p_i\dot{q}^i - \lambda H(t) + i\lambda'\bar{\mathcal{P}}\mathcal{P}\Big)\Big\}. \quad (6.6)$$

The delta functions imply that $\mathcal{P}$, $\bar{\mathcal{P}}$, and u and therefore also λ are constants, and that $t=\lambda\tau+c$ where c is another constant. Following the procedure of subsection 4.3 we define the physical propagator by

$$K(q'',q';t'',t')\equiv\int du'du''d\mathcal{P}'d\bar{\mathcal{P}}'d\mathcal{P}''d\bar{\mathcal{P}}''\langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^{*}\rangle. \tag{6.7}$$

Inserting (6.6) and performing the integration over $u,\mathcal{P}$ and $\bar{\mathcal{P}}$ yield then

$$K(q'',q';t'',t')=\varepsilon\langle q'',t''|q',t'\rangle,\quad \varepsilon=\mathrm{sign}((\tau''-\tau')\lambda'), \tag{6.8}$$

where the right-hand side is given by (6.1). (Since we are integrating over one more u than t in the time-slice procedure, also one delta function from $\delta(\lambda-\dot{t})$ contributes in the u-integrations which together with the $\mathcal{P},\bar{\mathcal{P}}$-integrations yields the sign factor.) Notice that $t''-t'=(\tau''-\tau')\lambda$. The result (6.8) is exactly the same as (4.38) which is expected since the treatment here is equivalent to the one in subsection 4.3. Notice that the propagator contains all information including whether or not we have positive normed states.

6.1 The general propagator

The gauge fixing fermion (4.1) is relevant for $|\phi\rangle$-states satisfying the conditions (3.5) and (3.24). It is more interesting to choose a $|\phi\rangle$-independent gauge fixing like (3.16), *i.e.*

$$\psi=\mathcal{P}\Lambda(V)+\bar{\mathcal{C}}\chi(T). \tag{6.9}$$

The singlet $|s\rangle=e^{[Q,\psi]}|\phi\rangle$ is the same as (4.1) provided $|\phi\rangle$ satisfies (3.5) and (3.24), and provided $\Lambda(V)$ and $\chi(T)$ are linear. However, this is not the case for the propagators. Choosing the same representation as above we find the following propagator for the gauge fixing fermion (6.9)

$$\langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^{*}\rangle=\int_{Path}\prod_{\tau}\frac{d^nqdtdud\bar{\mathcal{C}}d\mathcal{C}d^npd\pi d\pi_ud\mathcal{P}d\bar{\mathcal{P}}}{(2\pi)^{n+2}}\exp\{i\int_{\tau'}^{\tau''}d\tau L_{eff}(\tau)\},$$
$$L_{eff}(\tau)\equiv p_i\dot{q}^i+\pi\dot{t}+\pi_u\dot{u}+i\mathcal{P}\dot{\mathcal{C}}+i\bar{\mathcal{P}}\dot{\bar{\mathcal{C}}}-\lambda(\pi+H(t))+i\lambda'\bar{\mathcal{P}}\mathcal{P}-\pi_u\chi(t)-i\dot{\chi}(t)\bar{\mathcal{C}}\mathcal{C}, \tag{6.10}$$

where as before $R'\equiv\{t',iu',\mathcal{P}',i\bar{\mathcal{P}}',q'^i\}$ etc. and $\lambda(u)\equiv i\Lambda(iu)$. Defining the physical propagator as in (6.7) we find after integrations over the ghosts $\mathcal{P}$, $\bar{\mathcal{P}}$, and π, π_u

$$K(q'',q';t'',t')=\int du'du''\int_{Path}\prod_{\tau}\frac{d^nqdtdud^npd\bar{\mathcal{C}}d\mathcal{C}\rho(\lambda')}{(2\pi)^n}\delta(\dot{u}-\chi(t))\delta(\lambda(u)-\dot{t})\times$$
$$\exp\Big\{i\int_{\tau'}^{\tau''}d\tau\Big(p_i\dot{q}^i+i\frac{1}{\lambda'}\dot{\mathcal{C}}\dot{\bar{\mathcal{C}}}-\lambda H(t)-i\dot{\chi}(t)\bar{\mathcal{C}}\mathcal{C}\Big)\Big\}, \tag{6.11}$$

where $\rho(\lambda')=\lambda'$ is a measure factor. The conditions

$$\dot{u}=\chi(t),\quad \dot{t}=\lambda(u) \tag{6.12}$$

have well defined solutions if $\dot{\chi}(t)$ and $\lambda'(u)$ have definite signs which we require. Let us for simplicity consider the linear choice

$$\chi(t)=\alpha(t-t_0),\quad \lambda(u)=\beta u, \tag{6.13}$$

where α and β are real constants. The equations (6.12) have then the solutions

$$u(\tau) = Ae^{c\tau} + Be^{-c\tau}, \quad c \equiv \sqrt{\alpha\beta},$$
$$t(\tau) = t_0 + \frac{c}{\alpha}\Big(Ae^{c\tau} - Be^{-c\tau}\Big), \tag{6.14}$$

where A and B are constants. We have real solutions for arbitrary real A and B if $\alpha\beta > 0$ and for real $A = B$ or imaginary $A = -B$ if $\alpha\beta < 0$. For the linear choice (6.13) it is straight-forward to perform the integrations in u and $\mathcal{C}$, $\bar{\mathcal{C}}$ in (6.11). We find then exactly the solution (6.8). This result is what we should expect since the gauge fixing fermion (6.9) yields exactly the same singlets as $\psi = \mathcal{P}\Lambda(v)$ for linear $\chi(t)$ and $\Lambda(v)$ as was mentioned above. Thus, the physical wave functions are the same which means that the physical propagators should be the same. However, still the result is somewhat unexpected since the $|\phi\rangle$-states themselves do not enter the definition of the physical propagator. Only the conditions on $|\phi\rangle$ enter in the form of boundary conditions. (In fact, the solution (6.8) is also valid for nonlinear Λ and χ.)

Now the gauge fixing fermion (6.9) is good for any $|\phi\rangle$-state. In particular it is good for the $|\phi\rangle$-state considered in section 5. A corresponding physical propagator must be consistent with the conditions (5.6) on $|\phi\rangle$. This means that we should define the physical propagator as follows

$$K(q'', q'; u'', u') = \int dt' dt'' \langle R''| e^{(\tau''-\tau')[Q,\psi]} |R'^*\rangle\Big|_{\mathcal{P}'=\mathcal{P}''=\bar{\mathcal{P}}'=\bar{\mathcal{P}}''=0}. \tag{6.15}$$

Integrations over $\mathcal{C}, \bar{\mathcal{C}}$ and π, π_u yields ($\rho(\dot{\chi}) = \dot{\chi}$ is a measure factor)

$$K(q'', q'; u'', u') = \int dt' dt'' \int_{Path} \prod_{\tau} \frac{d^n q dt du d^n p d\mathcal{P} d\bar{\mathcal{P}} \rho(\lambda')}{(2\pi)^n} \delta(\dot{u} - \chi(t))\delta(\lambda(u) - \dot{t}) \times$$
$$\exp\Big\{ i \int_{\tau'}^{\tau''} d\tau \Big(p_i \dot{q}^i - i\frac{1}{\lambda'}\dot{\mathcal{P}}\dot{\bar{\mathcal{P}}} - \lambda H(t) + i\lambda'\bar{\mathcal{P}}\mathcal{P}\Big)\Big\}. \tag{6.16}$$

and a further integration over t and $\mathcal{P}, \bar{\mathcal{P}}$ yields then

$$K(q'', q'; u'', u') = \eta\langle q'', u''|q', u'\rangle, \quad \eta = \text{sign}((\tau'' - \tau')\dot{\chi}), \tag{6.17}$$

where $\langle q'', u''|q', u'\rangle$ with u replaced by t is given by (6.1) which is the appropriate physical propagator.

6.2 Remarks on interpretations

In all propagators above there appear delta functions implying the gauge fixing

$$\frac{dt}{d\tau} = \lambda(u). \tag{6.18}$$

Since the sign of $\tau'' - \tau'$ is fixed by construction we have sign$\lambda = \text{sign}(t'' - t')$ for $\tau'' > \tau'$ in agreement with previous results. However, if we for some reason restrict the sign of λ then $\text{sign}(\tau'' - \tau') \propto \text{sign}(t'' - t')$ and

$$K(q'', q'; t'', t') \propto \text{sign}(t'' - t')\langle q'', t''|q', t'\rangle, \tag{6.19}$$

which implies that K no longer satisfies the Schrödinger equation. On the other hand such a restriction also implies that we consider both positive and negative metric states for the BRST singlets. In order to restrict ourselves to positive metric states we should choose positive sign of ε which here means that K only is valid for $t'' > t'$ or $t'' < t'$ in which case the Schrödinger equation is satisfied.

6.3 Comparison with the standard BFV formulation

The standard BFV path integral formulation of the propagator is given by the right-hand side of (6.4) with $[Q, \psi]_W$ replaced by the Poisson bracket $\{Q, \psi\}$. In this formulation we obtain exactly the right-hand side of (6.10) for the choice $\psi = \mathcal{P}\lambda(u) + \bar{\mathcal{C}}\chi(t)$. Note that in the standard BFV prescription one should use real classical variables throughout in which case one always obtain a real effective Lagrangian. Let us summarize the differences between the two approaches:
1) $\lambda(u)$ is different from the classical function $\Lambda(v)$ used in the operator approach. In the linear case we have *e.g.* $\lambda(u) = -\Lambda(u)$.
2) In the BFV formulation we may choose $\lambda(u-u_0)$ which is unnatural but possible within the operator approach. (It violates manifest reality of the norms.) This choice yields the same result as above: Projected propagators do not depend on u_0.
3) In the BFV formulation one often considers τ-dependent gauge fixing $\chi(t)$. For instance, $t = \tau$ is a natural gauge fixing in the classical theory. A τ-dependent $\chi(t)$ is very unnatural in the operator approach although it is in principle possible. In fact, (6.18) suggests *e.g.* that we should choose $t = \lambda\tau$ for λ constant. (A similar gauge fixing has been proposed for the relativistic particle [26].)
4) All the above statements concern the particular representation in which v and $\bar{\mathcal{C}}$ and their conjugate momenta are imaginary. In the operator approach we may also consider other representations with complex time. Such representations are unnatural but possible within the standard BFV formulation although there is no general prescription for them there. The argument for an imaginary time and Euclidian field theory as due to indefinite metric space, *i.e.* the argument used in the present operator approach, was in the literature first given in [27]. Here we have shown that only the representation with real time represent the BRST singlets and the original theory.

In the general case the operator approach may even lead to an effective Lagrangian which depends on $\hbar$. Such terms might appear in the Weyl symbol $[Q, \psi]_W$ and cannot be argued for within the standard BFV formulation. This seems therefore to constitute a crucial difference between the two approaches which has to be clarified.

7 Conclusions

In this paper we have performed a rather detailed investigation of the BRST quantization of a simple class of reparametrization invariant theories corresponding to quantum mechanical systems, which we called cohomological quantum mechanics. The procedure used was a very precise operator version of the BFV formulation on inner product spaces which prescribes the BRST singlets to be of the form $|s\rangle \equiv e^{[Q,\psi]}|\phi\rangle$ where ψ is an appropriate hermitian gauge fixing fermion and where $|\phi\rangle$ satisfies conditions whose possible form are precisely prescribed. There are gauge invariant conditions as well as ghost and gauge

fixings. They determine the boundary conditions in the wave function representation of $|s\rangle$ as well as in propagators. By means of these boundary conditions our analysis has led us to propose the following general projection prescriptions:

1. Physical wave functions satisfying the Dirac quantization are obtained from the wave function representations of $|s\rangle$ by imposing boundary conditions corresponding to the conditions on $|\phi\rangle$ except for the gauge fixings.

2. Gauge fixed physical wave functions, which not necessarily satisfy the Dirac quantization conditions, are obtained from the above projected wave function when the boundary conditions corresponding to the gauge fixings are imposed. They are also obtained from the wave function representations of $|\phi\rangle$ by imposing the dual or conjugated boundary conditions to the gauge fixings.

3. Physical propagators satisfying the Dirac quantization are obtained from the matrix representation of $e^{[Q,\psi]}$ for an appropriate gauge fixing fermion ψ, a representation which in general is equivalent to the conventional BFV path integral. One has then to impose the boundary conditions corresponding to the conditions on $|\phi\rangle$ except for the gauge fixings.

For cohomological quantum mechanics we verified these rules explicitly. The physical wave functions in 1 and propagators in 3 were shown to satisfy the original Schrödinger equation exactly what we should have. Property 3 was shown to be true for gauge fixing fermions appropriate for a particular set of conditions on $|\phi\rangle$, and for those that are independent of the conditions on $|\phi\rangle$. Their path integral expressions were shown to agree with the projections obtained from the standard BFV prescription. The corresponding path integrals represent therefore solutions to the Dirac quantization. In [25] this was proposed to be a general property for a particular set of boundary conditions. (This was generalized to reducible gauge theories in [29].) Our results suggest that this is a general property for all allowed boundary conditions. The understanding of these boundary conditions is also made more precise. (Reducible theories are considered in [16].)

Our projected wave functions may also depend on complex time since the reality property of time is just a choice of representation. However, we have found that only wave functions for real time represent the BRST singlets. Although we expect that our projection rules are valid in general we need to verify this for many more models. In the case of the ordinary relativistic particle the rule 1 yields wave functions that satisfy the Klein-Gordon equation. This may easily be extracted from [30]. That the corresponding propagators satisfy 3 is verified in appendix C.

We hope that our treatment has further elucidated how physical time appears in a BRST quantization of a reparametrization invariant theory. Of course in gravity and other more complicated theories everything is much more complex since we then also have different topological sectors to deal with.

A Classical gauge invariant extensions

Consider a classical gauge theory in which ψ_s, $s = 1, \ldots, m$, are the gauge generators satisfying a Lie algebra. ($\psi_s = 0$ are first class constraints.) To every dynamical variable, A, there exists a gauge invariant extension in the gauge, $\chi^r = 0$, $r = 1, \ldots, m$, given by (formula (3.13) in [28])

$$A_{(\chi)} = \int d\Omega \, |\det\{\chi^r(\Omega), \psi_s(\Omega)\}| \delta^m(\chi(\Omega)) A(\Omega), \tag{A.1}$$

where $A(\Omega), \chi^r(\Omega)$ are the gauge transformed variable and gauge fixing conditions, which explicitly depend on the group coordinates. The integration is over the group volume.

In the cohomological models in the text there is only one gauge generator, $\pi + H$, and only one group coordinate. A gauge transformed dynamical variable A is here given by

$$A(u) \equiv e^{-uAd(\pi+H(t))} A = e^{u(\partial_t - adH - (\partial_t H)\partial_\pi)} A, \tag{A.2}$$

where

$$\begin{aligned} AdH &\equiv \{H, \cdot\} = adH + \frac{\partial H}{\partial t}\frac{\partial}{\partial \pi}, \\ adH &\equiv \{H, \cdot\}_{red} = \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}, \end{aligned} \tag{A.3}$$

where in turn $\{\ ,\ \}$ is the Poisson bracket and $\{\ ,\ \}_{red}$ the reduced Poisson bracket in which t and π are not dynamical. (It is *not* the Dirac bracket.)

Let now $\chi(t) = 0$ be gauge fixing conditions that fix t to be t_0 as in the text. From (A.1) we have then that the gauge invariant extension $A_{(\chi)}$ of any dynamical variable A is given by ($\pi + H(t)$ is invariant under group transformations here)

$$\begin{aligned} A_{(\chi)} &\equiv \int du \, |\det\{\chi(t+u), \pi + H(t)\}| \delta(\chi(t+u)) A(u) = \\ &= e^{-u(Ad\pi + AdH)} A\Big|_{u=t_0-t}. \end{aligned} \tag{A.4}$$

It is easy to see that $A_{(\chi)}$ is gauge invariant;

$$(Ad\pi + AdH(t)) A_{(\chi)} = \{\pi + H(t), A_{(\chi)}\} = 0. \tag{A.5}$$

(Take *e.g.* the derivative of $A_{(\chi)}$ with respect to t_0 using the formal expression (A.4). One get then $(Ad\pi + AdH)A_{(\chi)}$ plus the derivative of $A_{(\chi)}$ with respect to t_0 again.) Eq.(A.5) may be interpreted as follows: If we view $H(t)$ as the Hamiltonian then we have

$$\frac{dA_{(\chi)}}{dt} = \frac{\partial A_{(\chi)}}{\partial t} + \{A_{(\chi)}, H\} = -\{\pi + H, A_{(\chi)}\} = 0, \tag{A.6}$$

which means that the gauge invariant extensions may be viewed as the constants of motion with respect to t.

We want now to derive the equations of motion with respect to t_0. We need then the following gauge invariant extensions

$$\bar{q}^i(t_0) \equiv q^i_{(\chi)} \equiv \int du \, |\det\{\chi(t+u), \pi + H(t)\}| \delta(\chi(t+u)) q^i(u) =$$

$$
\begin{aligned}
&= e^{u(\partial_t - adH(t))} q^i \Big|_{u=t_0-t}, \\
&\bar{p}_i(t_0) \equiv p_{i(\chi)} \equiv \int du \, |\det\{\chi(t+u), \pi + H(t)\}| \delta(\chi(t+u)) p_i(u) = \\
&= e^{u(\partial_t - adH(t))} p_i \Big|_{u=t_0-t}, \\
&\bar{H}(t_0) \equiv H_{(\chi)} \equiv \int du \, |\det\{\chi(t+u), \pi + H(t)\}| \delta(\chi(t+u)) H(t,u) = \\
&= e^{u(\partial_t - adH(t))} H(t) \Big|_{u=t_0-t}, \\
&t_{(\chi)} \equiv \int du \, |\det\{\chi(t+u), \pi + H(t)\}| \delta(\chi(t+u))(t+u) = t_0.
\end{aligned} \tag{A.7}
$$

The last relation states that the gauge invariant extension of time is t_0, which means that it is correct to look for the equations with respect to this parameter. Now the gauge invariant extensions are just the canonical transformations of the original variables. This means that

$$
\bar{H}(t_0) = H(\bar{q}, \bar{p}, t_0). \tag{A.8}
$$

where $H(q,p,t)$ is the original Hamiltonian. Notice also that $\bar{q}^i(t_0)\big|_{t_0=t} = q^i$, $\bar{p}_i(t_0)|_{t_0=t} = p_i$, and $\bar{H}(t_0)\big|_{t_0=t} = H$. We have now Hamilton's equations

$$
\begin{aligned}
\frac{d\bar{q}^i(t_0)}{dt_0} &= \{\bar{q}^i(t_0), \bar{H}(t_0)\}, \\
\frac{d\bar{p}_i(t_0)}{dt_0} &= \{\bar{p}_i(t_0), \bar{H}(t_0)\}.
\end{aligned} \tag{A.9}
$$

Proof:

$$
\begin{aligned}
\frac{d\bar{q}^i(t_0)}{dt_0} &= e^{u(\partial_t - adH(t))}(\partial_t - adH(t)) q^i \Big|_{u=t_0-t} = e^{u(\partial_t - adH(t))} \{q^i, H(t)\} \Big|_{u=t_0-t} = \\
&= \{e^{u(\partial_t - adH(t))} q^i, e^{u(\partial_t - adH(t))} H(t)\} \Big|_{u=t_0-t} = \{\bar{q}^i(t_0), \bar{H}(t_0)\}.
\end{aligned} \tag{A.10}
$$

In the special case when H has no explicit time dependence we have in particular

$$
\begin{aligned}
&\bar{H}(t_0) = H(\bar{q}, \bar{p}) = H(q,p), \quad \bar{q}^i(t_0) = e^{-(t_0-t)adH} q^i, \\
&\bar{p}_i(t_0) = e^{-(t_0-t)adH} p_i,
\end{aligned} \tag{A.11}
$$

from which (A.9) follows trivially. Eq.(A.9) are the correct classical Hamilton's equations since the Poisson bracket may be expressed in terms of the canonical coordinates $\bar{q}^i$, $\bar{p}_i$.

B Properties of our ghost states

Following the conventions of [23] we define ghost eigenstates as follows (we put hats on the operators here for clarity)

$$
\begin{aligned}
|\mathcal{C}, \bar{\mathcal{C}}\rangle &\equiv e^{-\mathcal{C}\hat{\mathcal{P}} - \bar{\mathcal{C}}\hat{\bar{\mathcal{P}}}} |0\rangle_{\mathcal{C}\bar{\mathcal{C}}}, \\
|\mathcal{P}, \bar{\mathcal{P}}\rangle &\equiv e^{-\mathcal{P}\hat{\mathcal{C}} - \bar{\mathcal{P}}\hat{\bar{\mathcal{C}}}} |0\rangle_{\mathcal{P}\bar{\mathcal{P}}},
\end{aligned} \tag{B.1}
$$

where the vacuum states are normalized as follows

$$
\begin{aligned}
&{}_{\mathcal{P}\bar{\mathcal{P}}}\langle 0|0\rangle_{\mathcal{C},\bar{\mathcal{C}}} = {}_{\mathcal{C},\bar{\mathcal{C}}}\langle 0|0\rangle_{\mathcal{P}\bar{\mathcal{P}}} = 1,\\
&{}_{\mathcal{P}\bar{\mathcal{P}}}\langle 0|\hat{\mathcal{C}}\hat{\bar{\mathcal{C}}}|0\rangle_{\mathcal{P}\bar{\mathcal{P}}} = {}_{\mathcal{C},\bar{\mathcal{C}}}\langle 0|\hat{\mathcal{P}}\hat{\bar{\mathcal{P}}}|0\rangle_{\mathcal{C},\bar{\mathcal{C}}} = i.
\end{aligned} \tag{B.2}
$$

This implies ($\langle \mathcal{C}^*, \bar{\mathcal{C}}^*| \equiv (|\mathcal{C}, \bar{\mathcal{C}}\rangle)^\dagger$)

$$
\begin{aligned}
&\langle \mathcal{C}^*, \bar{\mathcal{C}}^*|\mathcal{C}', \bar{\mathcal{C}}'\rangle = i\delta(\mathcal{C}^* - \mathcal{C}')\delta(\bar{\mathcal{C}}^* - \bar{\mathcal{C}}'),\\
&\langle \mathcal{C}^*, \bar{\mathcal{C}}^*|\mathcal{P}, \bar{\mathcal{P}}\rangle = e^{-\mathcal{P}\mathcal{C}^* - \bar{\mathcal{P}}\bar{\mathcal{C}}^*}.
\end{aligned} \tag{B.3}
$$

For $\mathcal{C}$ real and $\bar{\mathcal{C}}$ imaginary ($= \pm i\bar{\mathcal{C}}$) we have in particular

$$
\begin{aligned}
&\langle \mathcal{C}, i\bar{\mathcal{C}}|\mathcal{C}', i\bar{\mathcal{C}}'\rangle = \delta(\bar{\mathcal{C}} - \bar{\mathcal{C}}')\delta(\mathcal{C} - \mathcal{C}'),\\
&\langle \mathcal{C}, -i\bar{\mathcal{C}}|\mathcal{C}', -i\bar{\mathcal{C}}'\rangle = \delta(\mathcal{C} - \mathcal{C}')\delta(\bar{\mathcal{C}} - \bar{\mathcal{C}}'),
\end{aligned} \tag{B.4}
$$

where all coordinates are real. The corresponding completeness relations are

$$
\begin{aligned}
&\int |\mathcal{C}, i\bar{\mathcal{C}}\rangle d\mathcal{C} d\bar{\mathcal{C}} \langle \mathcal{C}, i\bar{\mathcal{C}}| = \mathbf{1},\\
&\int |\mathcal{C}, -i\bar{\mathcal{C}}\rangle d\bar{\mathcal{C}} d\mathcal{C} \langle \mathcal{C}, -i\bar{\mathcal{C}}| = \mathbf{1}.
\end{aligned} \tag{B.5}
$$

Note that $\int d\mathcal{C}\mathcal{C} = i$ and $\int d\bar{\mathcal{C}}\bar{\mathcal{C}} = i$ in the conventions we are using [23]. (For $\int d\mathcal{C}\mathcal{C} = \int d\bar{\mathcal{C}}\bar{\mathcal{C}} = 1$ one has to do the replacement $d\mathcal{C}d\bar{\mathcal{C}} \leftrightarrow d\bar{\mathcal{C}}d\mathcal{C}$.) The properties of $|\mathcal{P}, \pm i\bar{\mathcal{P}}\rangle$ are also given by (B.4) and (B.5) with $\mathcal{C}, \bar{\mathcal{C}}$ replaced by $\mathcal{P}, \bar{\mathcal{P}}$. Note that

$$
\langle \mathcal{C}, \pm i\bar{\mathcal{C}}|\mathcal{P}, \mp i\bar{\mathcal{P}}\rangle = e^{-\mathcal{P}\mathcal{C} - \bar{\mathcal{P}}\bar{\mathcal{C}}}, \tag{B.6}
$$

according to (B.3).

The Weyl transform A_W (or Weyl symbol) of an even operator A is defined by

$$
\begin{aligned}
&A(\hat{\mathcal{P}}, \hat{\bar{\mathcal{P}}}, \hat{\mathcal{C}}, \hat{\bar{\mathcal{C}}}) = \int d\beta_1 d\beta_2 d\alpha_1 d\alpha_2 A_W(\alpha_1, \alpha_2, \beta_1, \beta_2)\Delta(\alpha_1, \alpha_2, \beta_1, \beta_2),\\
&\Delta(\alpha_1, \alpha_2, \beta_1, \beta_2) \equiv\\
&\int d\lambda_2 d\lambda_1 d\xi_2 d\xi_1 \exp\{(\alpha_1 - \hat{\mathcal{P}})\lambda_1 + (\alpha_2 - \hat{\bar{\mathcal{P}}})\lambda_2 + (\beta_1 - \hat{\mathcal{C}})\xi_1 + (\beta_2 - \hat{\bar{\mathcal{C}}})\xi_2\}.
\end{aligned} \tag{B.7}
$$

From this definition we get (cf [20])

$$
\begin{aligned}
&\langle \mathcal{P}, i\bar{\mathcal{P}}|A(\hat{\mathcal{P}}, \hat{\bar{\mathcal{P}}}, \hat{\mathcal{C}}, \hat{\bar{\mathcal{C}}})|\mathcal{P}', i\bar{\mathcal{P}}'\rangle =\\
&\int d\bar{\mathcal{C}} d\mathcal{C} A_W(\frac{1}{2}(\mathcal{P} + \mathcal{P}'), i\frac{1}{2}(\bar{\mathcal{P}} + \bar{\mathcal{P}}'), \mathcal{C}, \bar{\mathcal{C}}) e^{-\mathcal{C}(\mathcal{P} - \mathcal{P}')} e^{-\bar{\mathcal{C}}(\bar{\mathcal{P}} - \bar{\mathcal{P}}')},\\
&\langle \mathcal{C}, -i\bar{\mathcal{C}}|A(\hat{\mathcal{P}}, \hat{\bar{\mathcal{P}}}, \hat{\mathcal{C}}, \hat{\bar{\mathcal{C}}})|\mathcal{C}', -i\bar{\mathcal{C}}'\rangle =\\
&\int d\mathcal{P} d\bar{\mathcal{P}} A_W(\mathcal{P}, \bar{\mathcal{P}}, \frac{1}{2}(\mathcal{C} + \mathcal{C}'), -i\frac{1}{2}(\bar{\mathcal{C}} + \bar{\mathcal{C}}')) e^{-\mathcal{P}(\mathcal{C} - \mathcal{C}')} e^{-\bar{\mathcal{P}}(\bar{\mathcal{C}} - \bar{\mathcal{C}}')}.
\end{aligned} \tag{B.8}
$$

These expressions determine the Weyl symbol $[Q, \psi]_W$ in the path integrals in section 6.

C Application to the relativistic particle

A free relativistic particle satisfies the mass shell condition

$$p^2 + m^2 = 0. \tag{C.1}$$

Efter quantization this constraint yields the Klein-Gordon equation. The appropriate BFV-BRST charge is then

$$Q = \frac{1}{2}(P^2 + m^2)\mathcal{C} + \Pi_V \bar{\mathcal{P}}, \tag{C.2}$$

which has the same form as for cohomological quantum mechanics in the text except that $\Pi + H$ is replaced by $\frac{1}{2}(P^2 + m^2)$. The canonical set of variables are (X^μ, P^μ), (V, Π_V), $(\mathcal{C}, \mathcal{P})$ and $(\bar{\mathcal{C}}, \bar{\mathcal{P}})$ which satisfy the commutator algebra (3.2) and (3.4) and

$$[X^\mu, P^\nu] = i\eta^{\mu\nu}, \tag{C.3}$$

where $\eta^{\mu\nu}$ is the Minkowski metric with signature $(-,+,+,+)$. As in cohomological quantum mechanics it is natural to use the representation in which V and $\bar{\mathcal{P}}$ have imaginary eigenvalues. In [30] and [15] two forms of the BRST singlets for the relativistic particle were given. Let $\psi_1(x, iu, \mathcal{P}, i\bar{\mathcal{P}})$ be the wave function representation of the singlet in [30] where iu and $i\bar{\mathcal{P}}$ are eigenvalues of the operators V and $\bar{\mathcal{P}}$. The appropriate boundary conditions are here $\pi_u = \mathcal{C} = \bar{\mathcal{C}} = 0$. It is then straight-forward to check that $\phi_1(x) \equiv \int dud\mathcal{P}d\bar{\mathcal{P}}\psi_1(x, iu, \mathcal{P}, i\bar{\mathcal{P}})$ satisfies the Klein-Gordon equation. Let $\psi_2(x, iu, \mathcal{P}, i\bar{\mathcal{P}})$ be the wave function representation of the singlet in [15]. The appropriate boundary conditions are then $p^2 + m^2 = \mathcal{P} = \bar{\mathcal{P}} = 0$. The projected wave function $\phi_2(x) \equiv \int dx^0 \psi_2(x, iu, \mathcal{P} = 0, i\bar{\mathcal{P}} = 0)$ satisfies then the Klein-Gordon equation where time x^0 may be identified with $a+u$ where the parameter a enters the chosen gauge fixing fermion which is $\psi = \bar{\mathcal{C}}(X^0 - a)$ here. To be precise we should define the physical wave function by

$$\tilde{\phi}_2(x) \equiv \int d^4p\delta(p^2 + m^2)e^{ip\cdot x}\tilde{\psi}_2(p, iu, \mathcal{P}, i\bar{\mathcal{P}}), \tag{C.4}$$

where $\tilde{\psi}_2$ is the Fourier transform of ψ_2 with respect to x^μ. Also $\tilde{\phi}_2(x)$ satisfies the Klein-Gordon equation. $\phi_2(x)$ and $\tilde{\phi}_2(x)$ are equivalent apart from a factor p^0 at $x^0 = a + u$.

Let us now consider the general propagator

$$\langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^*\rangle, \tag{C.5}$$

where Q is given by (C.2) and where the gauge fixing fermion ψ is

$$\psi = \alpha\mathcal{P}(V - v_0) + \beta\bar{\mathcal{C}}(X^0 - a), \tag{C.6}$$

where α, β, v_0 and a are real parameters. We find

$$[Q, \psi] = \frac{1}{2}\alpha(P^2 + m^2)(V - v_0) - i\alpha\bar{\mathcal{P}}\mathcal{P} + \beta\pi(X^0 - a) - i\beta\mathcal{C}\bar{\mathcal{C}}P^0. \tag{C.7}$$

If $R = \{x^\mu, iu, \mathcal{C}, -i\bar{\mathcal{C}}\}$ in (C.5) where iu and $-i\bar{\mathcal{C}}$ are eigenvalues of the operators V and $\bar{\mathcal{C}}$ then (C.5) has the path integral representation (6.10) where the effective Lagrangian is (choosing $v_0 = 0$ and the eigenvalues of Π_V to be $-i\pi_u$)

$$\begin{aligned} L_{eff}(\tau) \equiv\ & p_\mu\dot{x}^\mu + \pi_u\dot{u} + i\mathcal{P}\dot{\mathcal{C}} + i\bar{\mathcal{P}}\dot{\bar{\mathcal{C}}} + \frac{1}{2}\alpha(p^2 + m^2)u - \\ & -i\alpha\bar{\mathcal{P}}\mathcal{P} - \beta\pi_u(x^0 - a) + i\beta\mathcal{C}\bar{\mathcal{C}}p^0, \end{aligned} \tag{C.8}$$

which also may be obtained from the BFV path integral prescription. Let us first calculate the projection to the physical propagator for the boundary conditions $\pi_u = \mathcal{C} = \bar{\mathcal{C}} = 0$. We obtain ($\rho(\alpha) = \alpha$ is a measure factor)

$$
\begin{aligned}
&K(x'';x') \equiv \int du'du''\langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^*\rangle\Big|_{\bar{\mathcal{C}}'=\mathcal{C}'=\bar{\mathcal{C}}''=\mathcal{C}'=0} = \\
&= \int du'du'' \int_{Path} \prod_\tau \frac{d^4x du d^4p d\bar{\mathcal{C}} d\mathcal{C}\rho(\alpha)}{(2\pi)^4}\delta(\dot{u} - \beta(x^0 - a)) \times \\
&\exp\Big\{i\int_{\tau'}^{\tau''} d\tau\Big(p_\mu \dot{x}^\mu + i\frac{1}{\alpha}\dot{\mathcal{C}}\dot{\bar{\mathcal{C}}} + i\beta p^0 \mathcal{C}\bar{\mathcal{C}} + \frac{1}{2}\alpha(p^2+m^2)u\Big)\Big\}\Big|_{\bar{\mathcal{C}}'=\mathcal{C}'=\bar{\mathcal{C}}''=\mathcal{C}'=0} = \varepsilon P(x'';x'), \\
&\varepsilon \equiv \mathrm{sign}(\alpha(\tau''-\tau')), \quad P(x'';x') \equiv \int d^4p\delta(\frac{1}{2}(p^2+m^2))e^{ip_\mu(x''^\mu - x'^\mu)}.
\end{aligned}
\tag{C.9}
$$

If we instead impose the boundary conditons $p^2 + m^2 = \mathcal{P} = \bar{\mathcal{P}} = 0$ we find

$$
\begin{aligned}
&K'(x'';x') \equiv \int d^4p'd^4p''\delta(\frac{1}{2}(p''^2+m^2))\delta(\frac{1}{2}(p'^2+m^2))e^{ip''\cdot x'' - ip'\cdot x'} \times \\
&\langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^*\rangle\Big|_{\bar{\mathcal{P}}'=\mathcal{P}'=\bar{\mathcal{P}}''=\mathcal{P}'=0} = \int d^4p'd^4p''\delta(\frac{1}{2}(p''^2+m^2))\delta(\frac{1}{2}(p'^2+m^2)) \times \\
&e^{ip''\cdot x''-ip'\cdot x'} \int_{Path}\prod_\tau \frac{d^4x du d^4p d\mathcal{P} d\bar{\mathcal{P}}\rho(\beta p^0)}{(2\pi)^4}\delta(\dot{u} - \beta(x^0 - a)) \times \\
&\exp\Big\{i\int_{\tau'}^{\tau''} d\tau\Big(-\dot{p}_\mu x^\mu - i\frac{1}{\beta p^0}\dot{\mathcal{P}}\dot{\bar{\mathcal{P}}} - i\alpha\bar{\mathcal{P}}\mathcal{P} + \frac{1}{2}\alpha(p^2+m^2)u\Big)\Big\}\Big|_{\bar{\mathcal{P}}'=\mathcal{P}'=\bar{\mathcal{P}}''=\mathcal{P}'=0} = \\
&= \varepsilon' P'(x'';x'), \quad \varepsilon' \equiv \mathrm{sign}(\beta(\tau''-\tau')), \\
&P'(x'';x') \equiv \int d^4p\, \mathrm{sign}(p^0)\delta(\frac{1}{2}(p^2+m^2))e^{ip_\mu(x''^\mu - x'^\mu)}.
\end{aligned}
\tag{C.10}
$$

Notice that the negative energy contribution enters with wrong sign in $P'(x'';x')$ as compared to $P(x'';x')$ in (C.9). This means that the last projections cannot lead to positive normed states which is the case in (C.9). This is consistent with the results of [30] and [15]. Notice that both $P(x'';x')$ in (C.9) and $P'(x'';x')$ in (C.10) satisfy the Klein-Gordon equation.

We may also use a different state space representation in which X^0 and P^0 have imaginary eigenvalues instead of V and Π_V. We have then the general propagator (C.6) with $R = \{\mathbf{x}, ix^0, v, \mathcal{C}, -i\bar{\mathcal{C}}\}$ for which the path integral representation (6.10) is valid with the effective Lagrangian ($a = 0$ here)

$$
\begin{aligned}
&L_{E\,eff}(\tau) \equiv p_\mu \dot{x}^\mu + \pi_v \dot{v} + i\mathcal{P}\dot{\mathcal{C}} + i\bar{\mathcal{P}}\dot{\bar{\mathcal{C}}} - i\frac{1}{2}\alpha(p^2+m^2)(v - v_0) - \\
&-i\alpha\bar{\mathcal{P}}\mathcal{P} + \beta\pi_v x^0 - \beta\mathcal{C}\bar{\mathcal{C}}p^0,
\end{aligned}
\tag{C.11}
$$

where 'E' denotes Euclidean since the metric for x^μ and p^μ is Euclidean. This effective Lagrangian is not entire real and it cannot be obtained from the standard BFV path integral prescription. For the boundary conditions $\pi_v = \mathcal{C} = \bar{\mathcal{C}} = 0$ we get here the physical propagator

$$
K_E(x'';x') \equiv \int dv'dv''\langle R''|e^{(\tau''-\tau')[Q,\psi]}|R'^*\rangle\Big|_{\bar{\mathcal{C}}'=\mathcal{C}'=\bar{\mathcal{C}}''=\mathcal{C}'=0} =
$$

$$= \int dv'dv'' \int_{Path} \prod_\tau \frac{d^4x dv d^4p d\bar{\mathcal{C}} d\mathcal{C} \rho(\alpha)}{(2\pi)^4} \delta(\dot{v} + \beta(x^0 - a)) \times$$

$$\exp\left\{ i\int_{\tau'}^{\tau''} d\tau \Big(p_\mu \dot{x}^\mu - i\frac{1}{\alpha}\dot{\mathcal{C}}\dot{\bar{\mathcal{C}}} - \beta p^0 \mathcal{C}\bar{\mathcal{C}} - i\frac{1}{2}\alpha(p^2+m^2)(v-v_0)\Big)\right\}\Bigg|_{\bar{\mathcal{C}}'=\mathcal{C}'=\bar{\mathcal{C}}''=\mathcal{C}'=0} =$$

$$= \varepsilon P_E(x'';x'), \quad \varepsilon \equiv \mathrm{sign}(\alpha(\tau''-\tau')), \quad P_E(x'';x') \equiv 2\int d^4p \frac{e^{ip_\mu(x''^\mu - x'^\mu)}}{p^2+m^2}. \qquad \text{(C.12)}$$

This is only valid if the Lagrange multiplier has the range (v_0, ∞) for $\alpha(\tau''-\tau') < 0$, and $(-\infty, v_0)$ for $\alpha(\tau''-\tau') > 0$. (We cannot have infinite range and these ranges are natural.) Such a restriction of the Lagrange multiplier has to be done at the very beginning by choosing variables ω, p_ω defined by either $v = v_0 + e^\omega, p_v = e^{-\omega}p_\omega$ or $v = v_0 - e^\omega, p_v = -e^{-\omega}p_\omega$ where ω has infinite range. (Such parametrization of the relativistic particle was made in [31].) The restriction in the ranges for the Lagrange multiplier above seem a little ad hoc. One should therefore start from a geometric argument for why $v - v_0$ should be positive or negative [26, 32]. Then one finds that the Euclidean representation above must be used. In (C.12) the sign of ε is not directly related to positive or negative metric states. We may have either or depending on whether or not the wave functions of the BRST singlets are even or odd in p^0 [30]. The propagator $P_E(x'';x')$ is a Wick rotated Feynman propagator which does not satisfy the Klein-Gordon equation. The Klein-Gordon equation is only satisfied if we restrict the sign of $x''^0 - x'^0$ which in a way is what the interpretation in subsection 6.2 suggests us to do since the range of the Lagrange multiplier is restriced here. For positive or negative time differences we have then propagators for only one branch of the solutions of the the mass shell condition. Notice that in this Euclidean case we cannot impose the boundary conditions $p^2 + m^2 = \mathcal{P} = \bar{\mathcal{P}} = 0$ since the first equality then implies $p^\mu = m = 0$.

Contrary to the case for cohomological quantum mechanics we have obtained three different physical propagators here. Their difference seems to be entirely due to a different treatment of the negative energy solution whose appearance is due to a nontrivial topology.

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